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Social Network Analysis

A.Y. 23/24

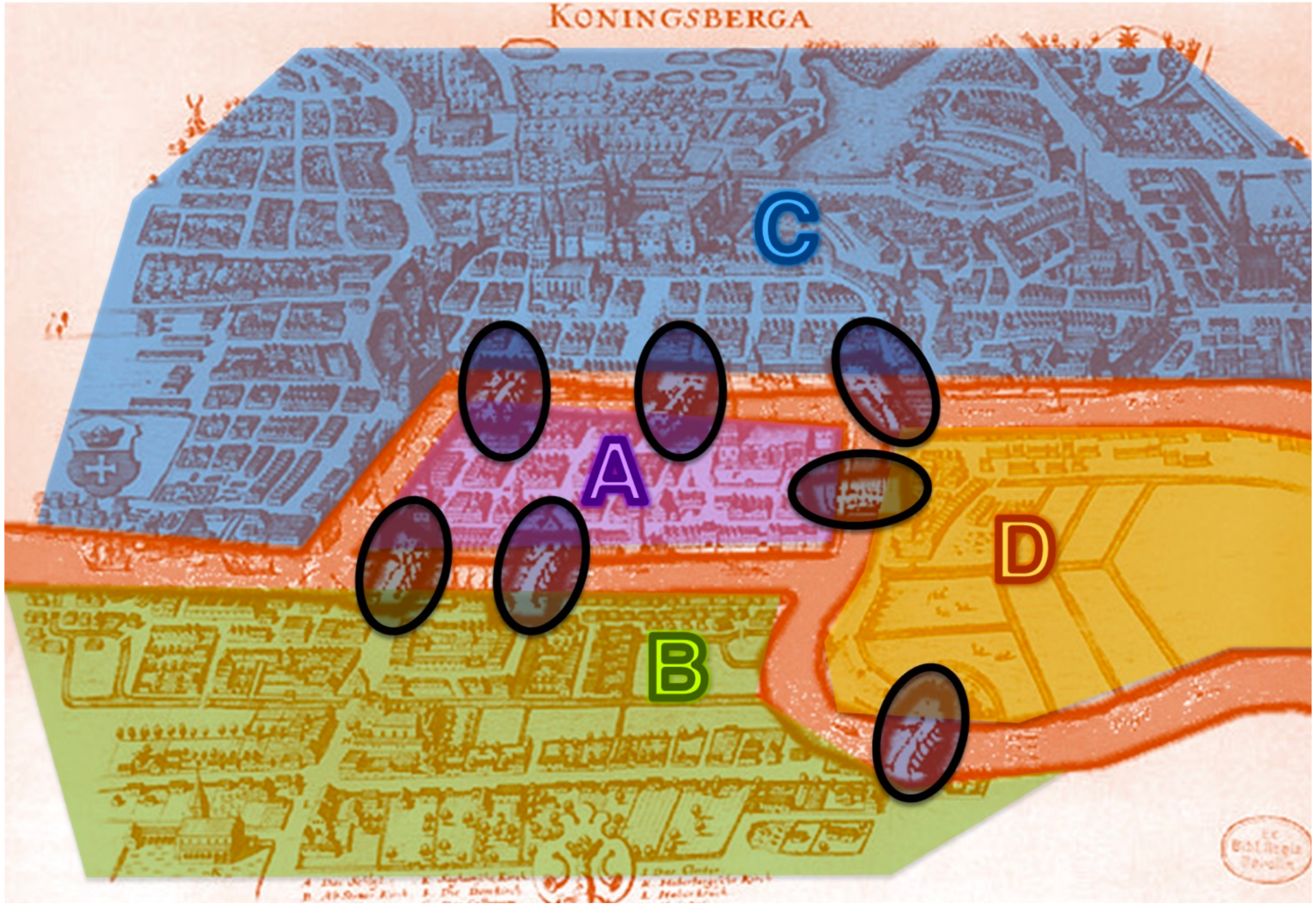
Communication Strategies

Graphs

an introduction

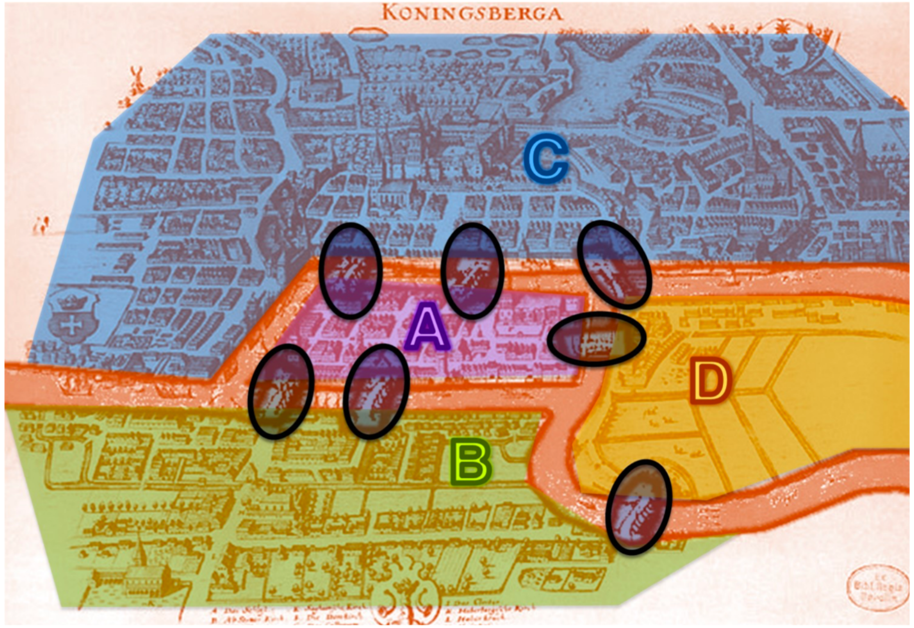
Euler and the 7 bridges of Könisberg

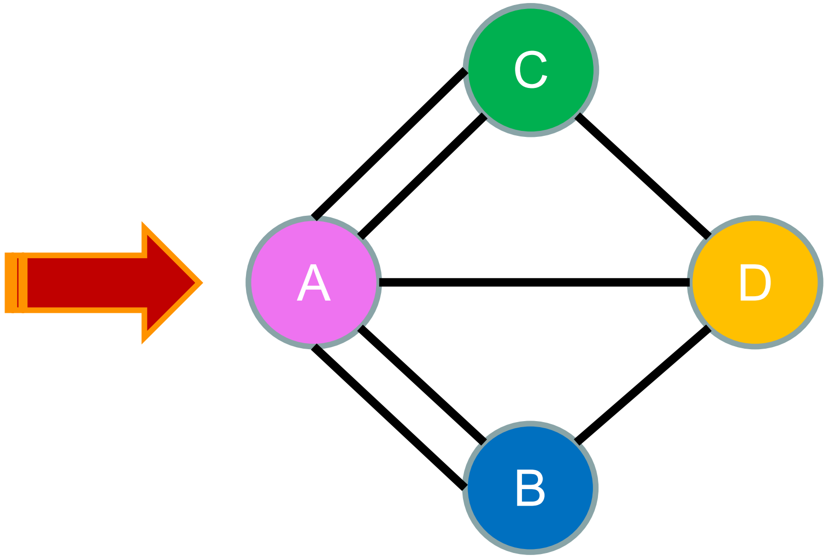
(Prussia, 1736) today Kaliningrad



How to walk through the city by crossing each bridge only once?

Networks as graphs





Graph $\mathcal{G}$ ($\mathcal{V}$,$\mathcal{E}$): network

- Vertices (set $\mathcal{V}$) : nodes, people, concepts
- Edges (set $\mathcal{E}$): links, relations, associations

mathematics *technology* *social psychology* *social cognition*

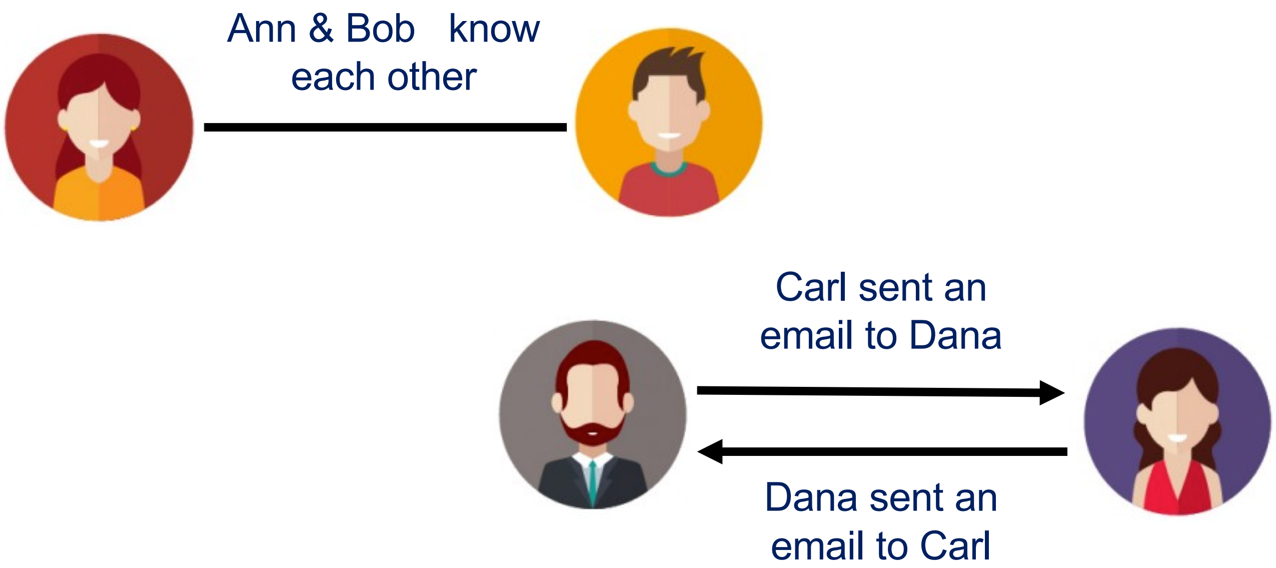
Directed versus undirected

- A connection relationship can have a privileged direction or can be mutual
  - Either a directed or an undirected link

- If the network has only (un)directed links, it is also called itself (un)directed network
  - Certain networks can have both types

Directed versus undirected

- At first glance undirected → directed by duplicating links, but not necessarily quite the same though



Some examples

network	nodes and links	type
Facebook	Profiles and friendship	undirected
Instagram	Accounts and followers	directed
the www	Webpages and links	directed
citation network	Papers and references	directed
social network	People and friends/acquaintances	undirected
movie network	Actors and co-starring	undirected
genealogy	People and parenthood	directed

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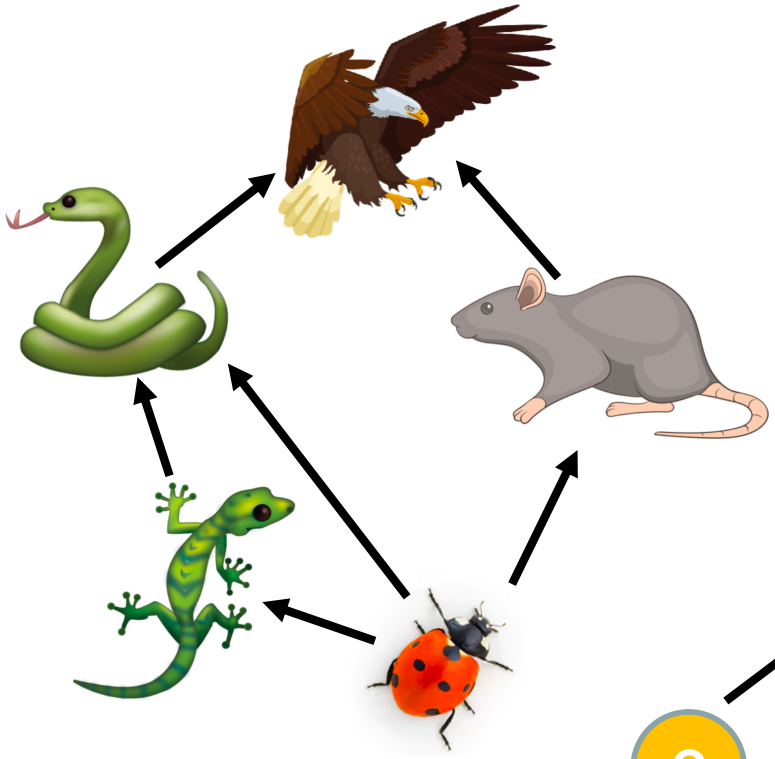
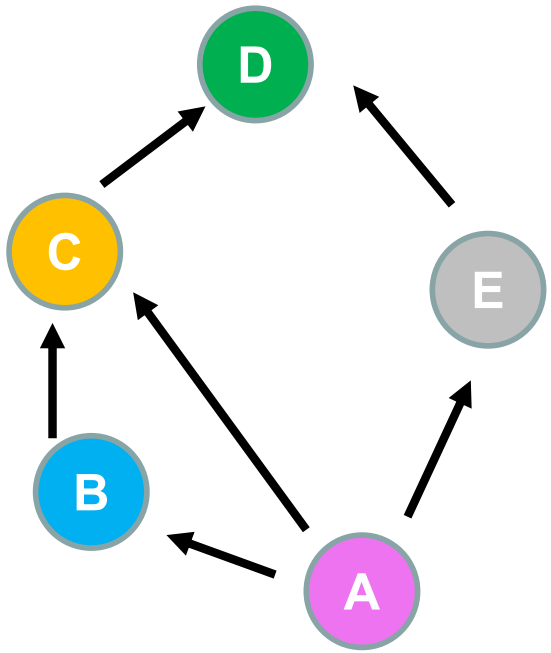
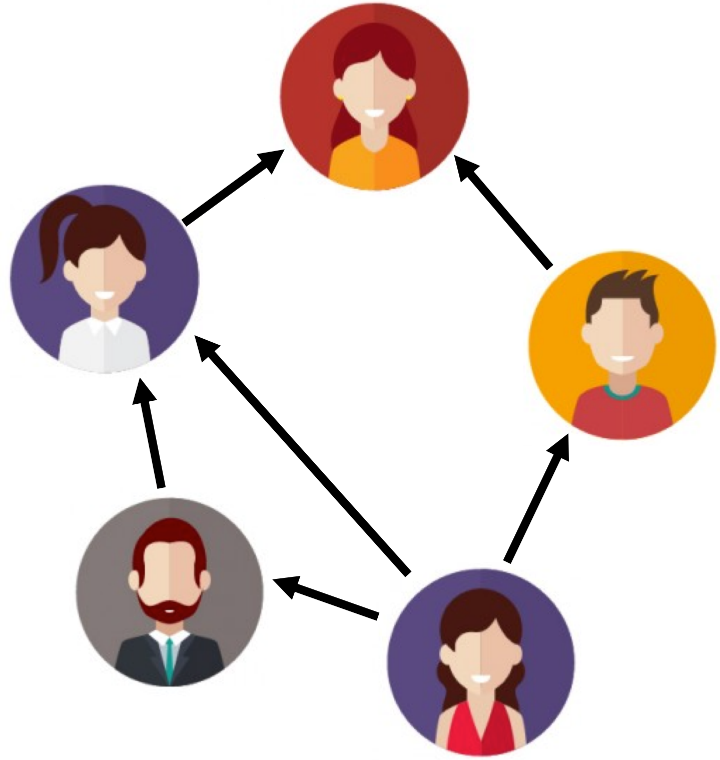
Can U think of other social networks?

network	nodes and links	type
Twitter	Accounts & follows	directed
WhatsApp	People & messages	directed
WhatsApp	People & contacts	undirected
TikTok	Accounts & friendship	undirected
LinkedIn	Acounts & friendship	undirected
TikTok	Accounts & follows	directed
Pinterest	People & image like	directed
YouTube	Accounts & followers	directed
YouTube	Accounts & collaborations	undirected
Ask	Accounts & replies	directed
LinkedIn	Acounts & followers	directed

Generality of representation





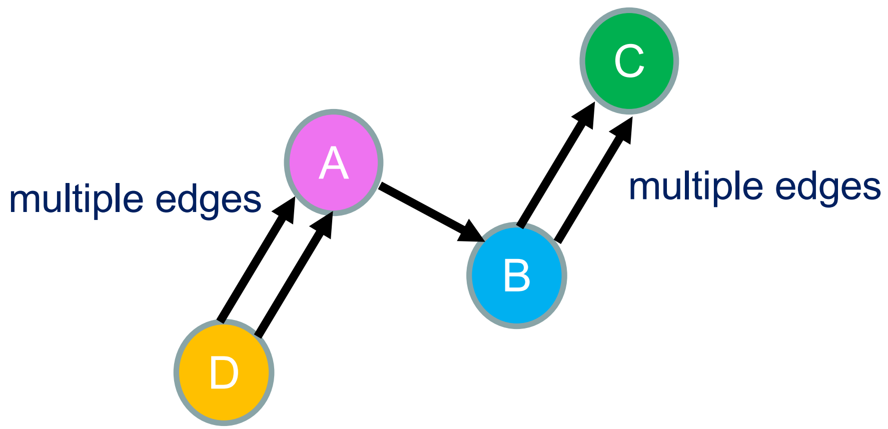


Graph representations

visual plot, adjacency mantix, edge list

Multi-graphs

- Multi-graphs (or pseudo-graphs)
  Some network representations require multiple links (e.g., number of citations from one author to another)

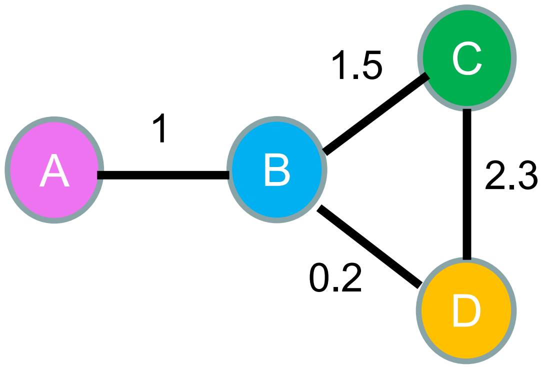


Weighted graphs

Weighted graph

Sometimes a weight is associated to a link, e.g., to underline that the links are not identical (strong/weak relationships)

Can be seen as a generalization of multi-graphs (weight = # of links)

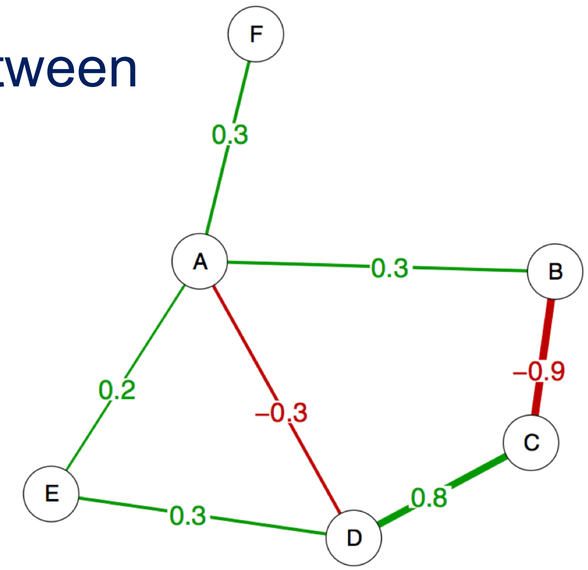


e.g., strength of a tie
- 0.2 = weak (acquaintances)
- 1 = strong (friends)
- 1.5 = stronger (close friends)
- 2.3 = very strong (best friends)

Signed graphs

- Edges can have signed values

  positive if there is an agreement between nodes
  negative if there's a disagreement



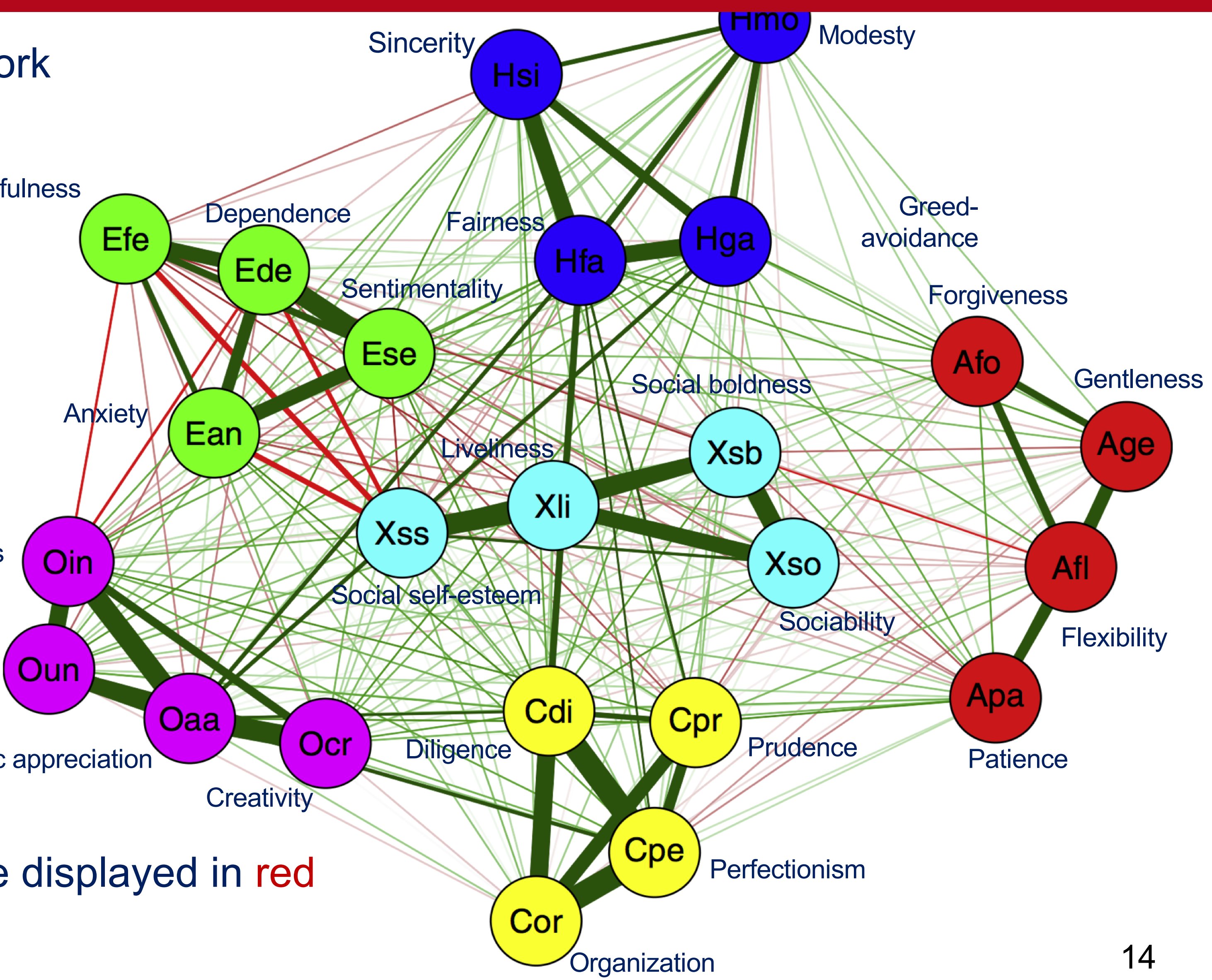
- This is typical of correlation networks

  correlation = a measure of similarity

- More difficult to handle

Signed graph example

A personality network
(Costantini et al, 2015)

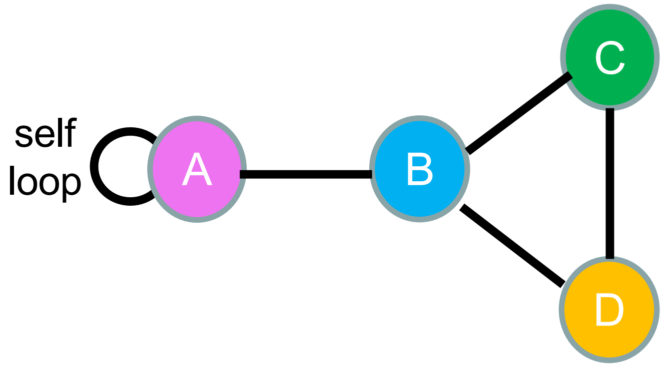


Negative links are displayed in red

Self interactions

- In many networks nodes do not interact with themselves

- To account for self-interactions, we add loops to represent them

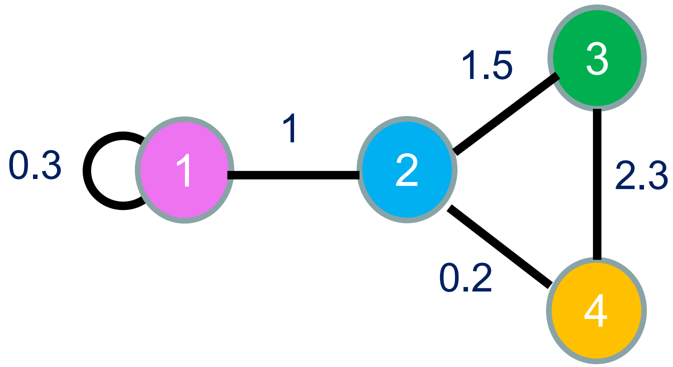


Adjacency matrix

- An adjacency matrix $A = [a_{ij}]$ associated to graph $\mathcal{G}$ has

  i is the row index — *j* is the column index

  entries $a_{ij} = 0$ if nodes *i* and *j* are not connected
  if nodes *i* and *j* are connected then $a_{ij} \neq 0$



this is a_{12}

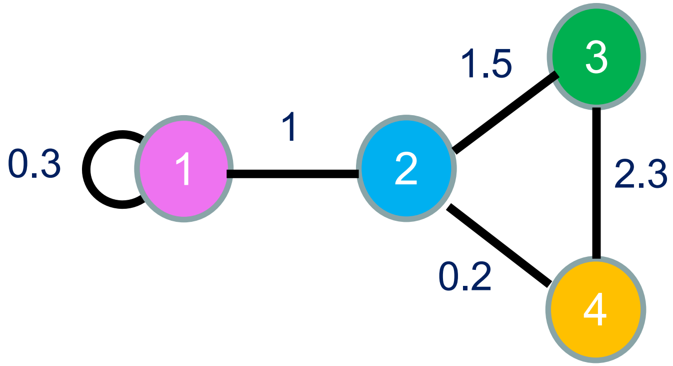
$$A = \begin{matrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0.2 \\ 0 & 1.5 & 0 & 2.3 \\ 0 & 0.2 & 2.3 & 0 \end{matrix}$$

← row 1

column 2

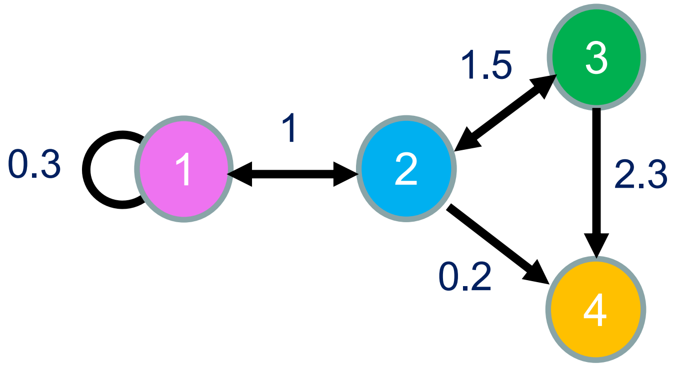
Symmetries

- Undirected graph = symmetric matrix



$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0.2 \\ 0 & 1.5 & 0 & 2.3 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

- Directed graph = asymmetric matrix



$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

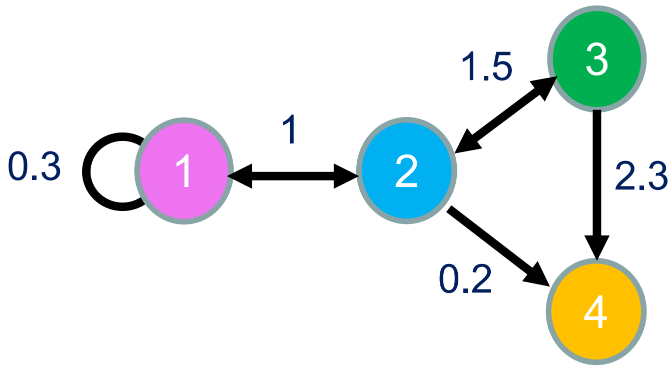
Convention

- The weight a_{ij} is associated to

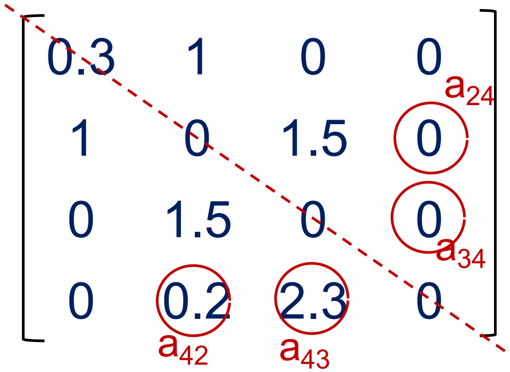
  i th row

  j th column

  directed edge $j \rightarrow i$ starting from node j and leading to node i



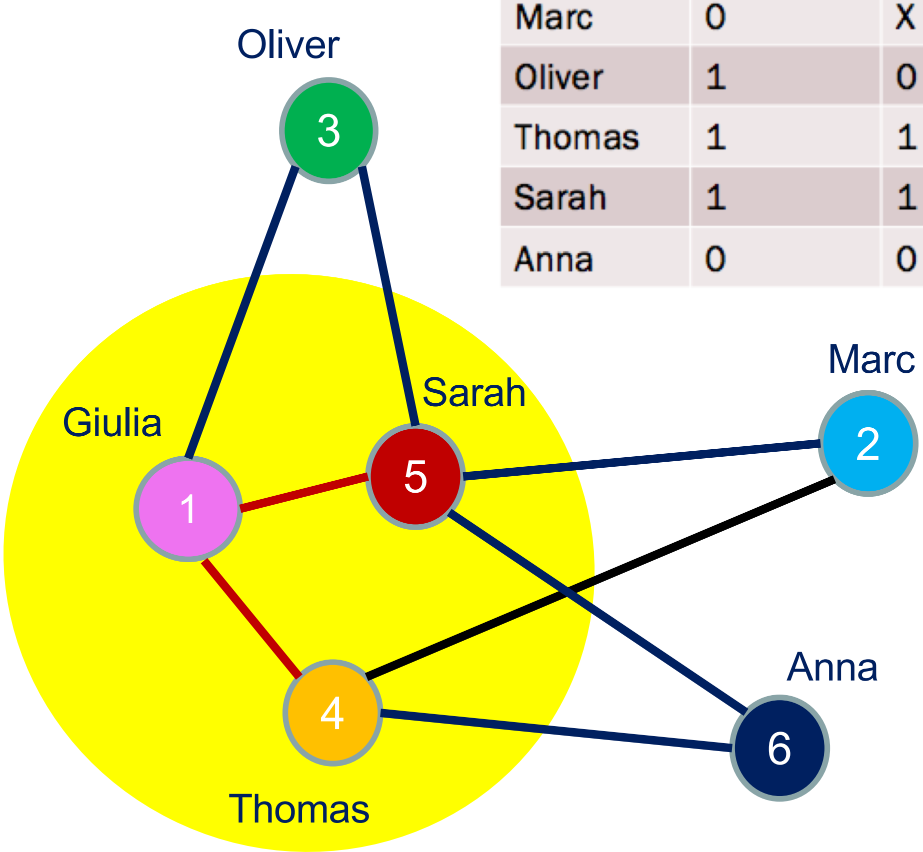
$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$



An example

which of these representations do you like best?

	Giulia	Marc	Oliver	Thomas	Sarah	Anna
Giulia	X					
Marc	0	X				
Oliver	1	0	X			
Thomas	1	1	0	X		
Sarah	1	1	1	0	X	
Anna	0	0	0	1	1	x



$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

which of these representations do you like best?

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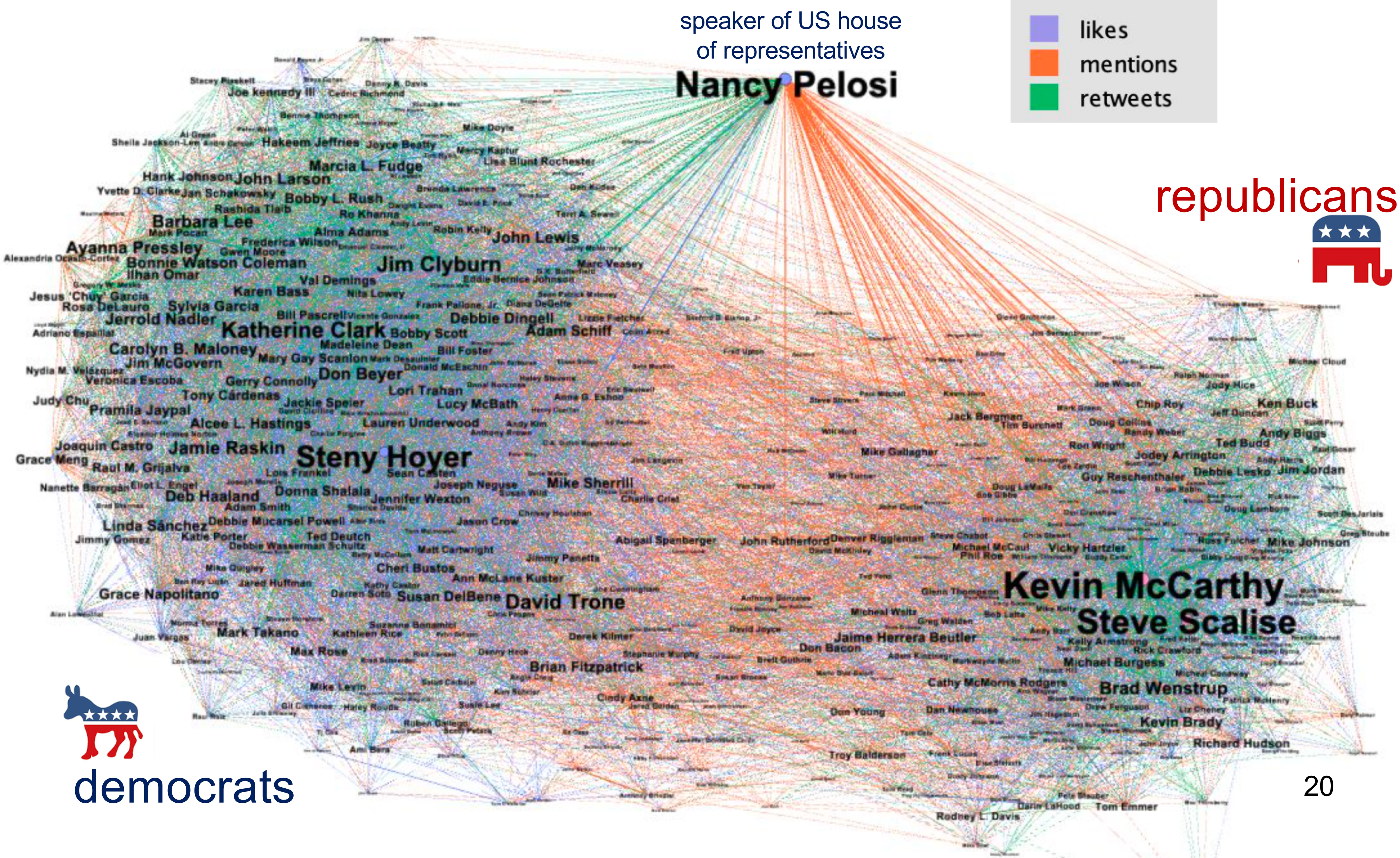
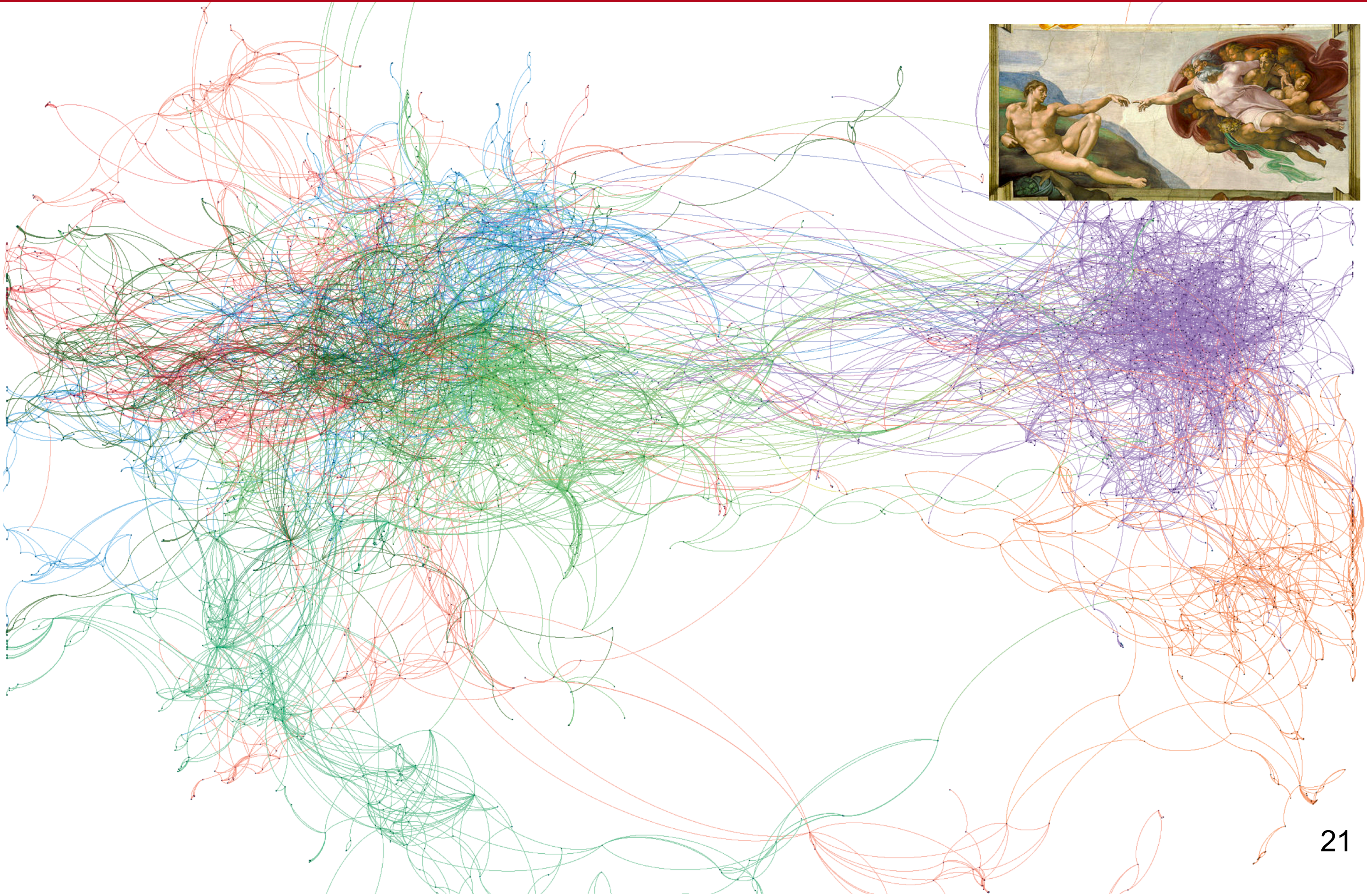
Graph plots may carry relevant info…

US republicans and democrats interactions on Twitter (2020)



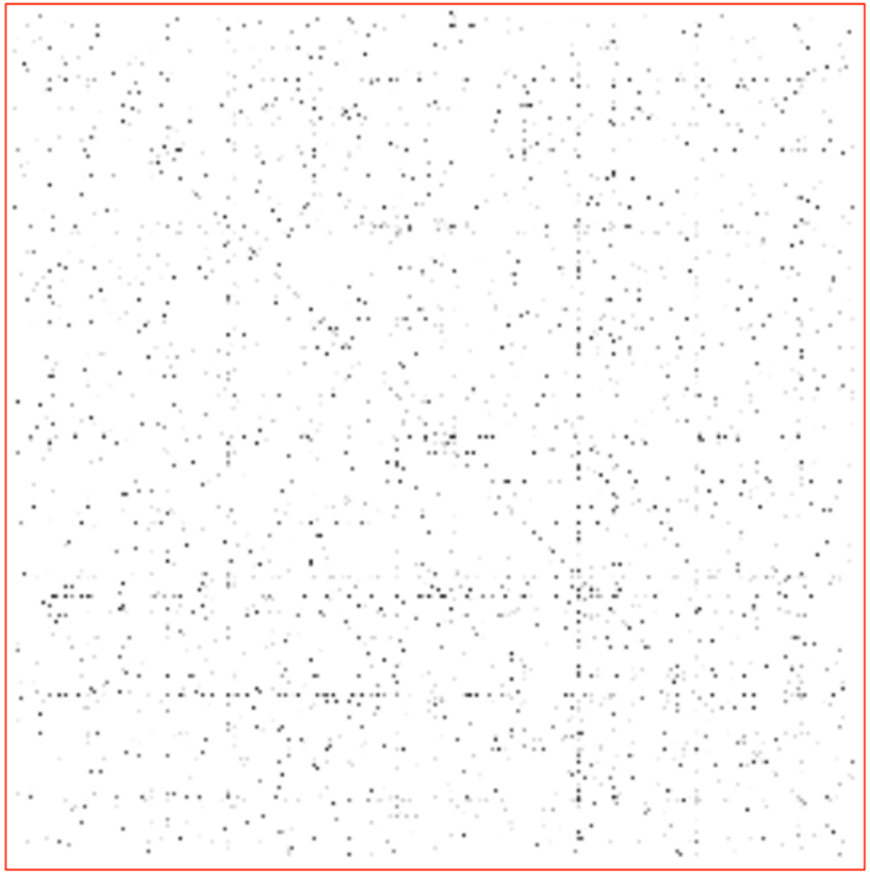
… or may not!



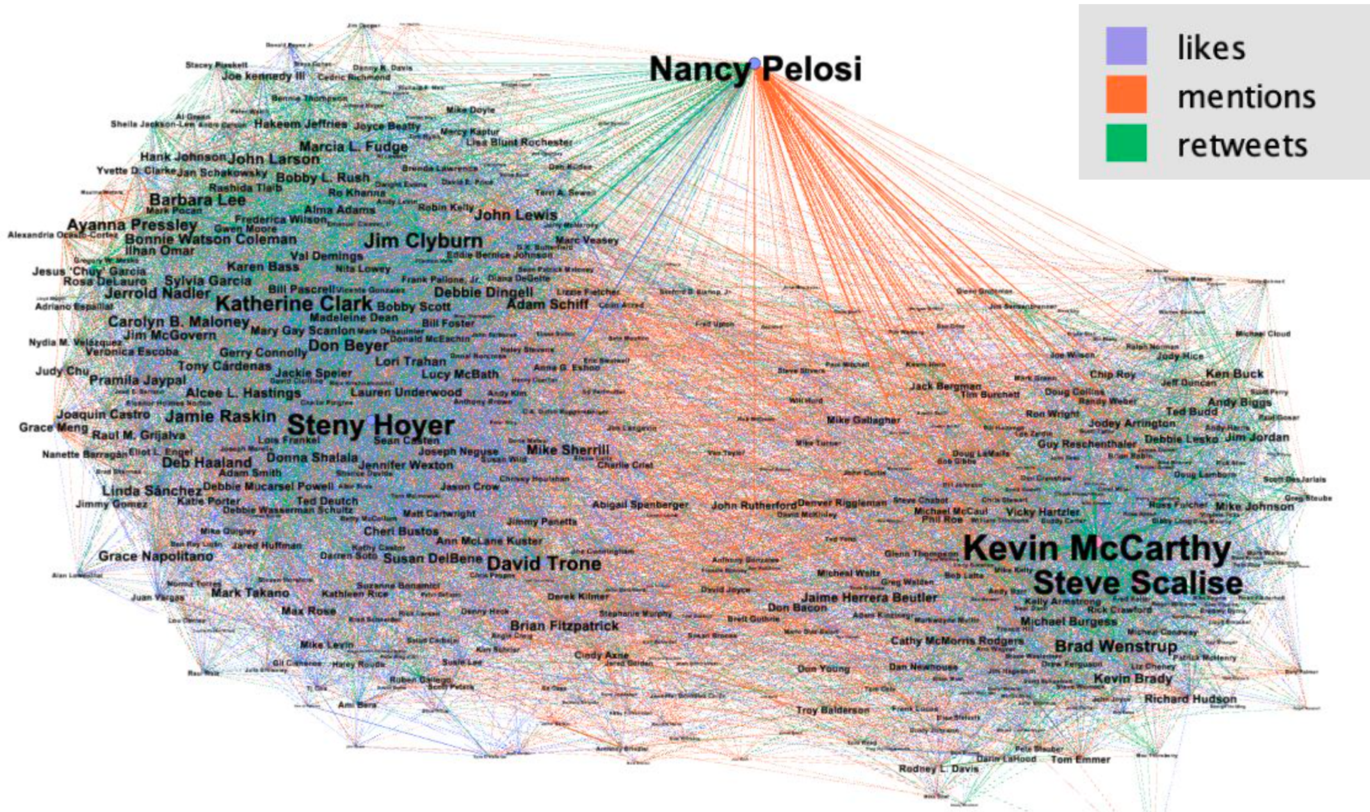
Real networks are sparse

- The adjacency matrix is typically sparse

good for tractability !

$A =$ 

Multi-layer networks



described by a set of adjacency matrices $\boldsymbol{A}_\ell$

e.g., one for likes, one for mentions, and one for retweets

A question 4 U

- So, what's the take-away so far?

Storing network data

adjacency matrix versus edge list

adjacency matrix — N^2 entries

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

edge list — L entries

1 → 3
1 → 4
1 → 5
2 → 4
2 → 5
3 → 5
4 → 6
5 → 6

Which one do U think is better?

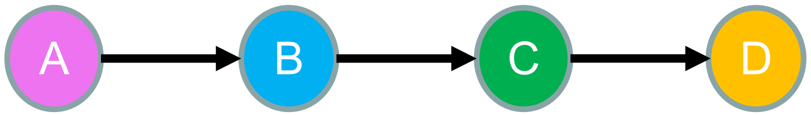
Distances in graphs

and related concepts

Paths

- **Path**

  a sequence of interconnected nodes (meaning each pair of nodes adjacent in the sequence are connected by a link)

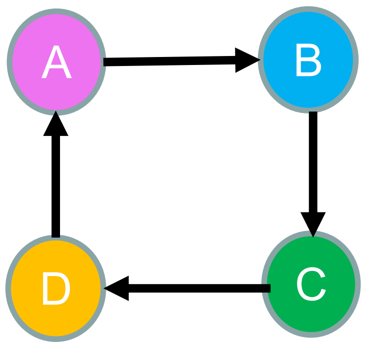


- **Path length**

  # of links involved in the path (if the path involves *n* nodes then the path link is *n-1*)

- **Cycle**

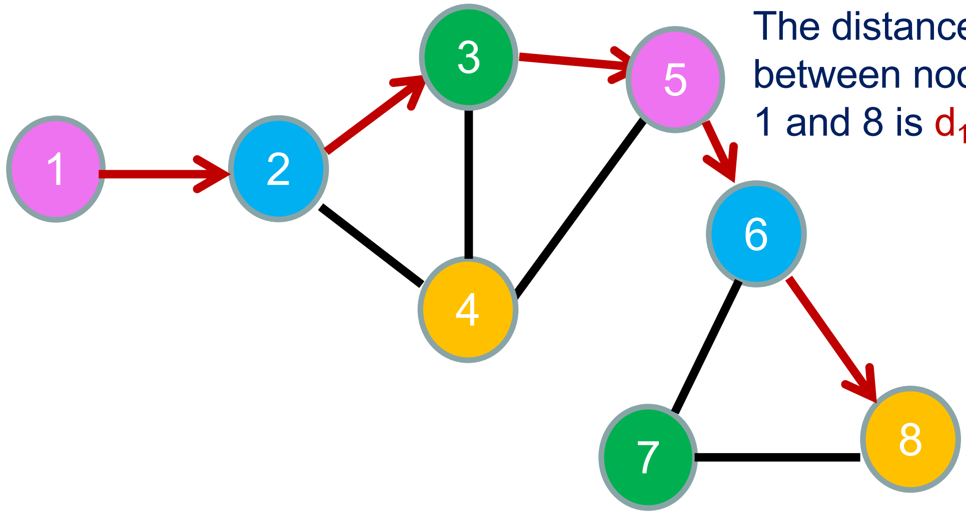
  path where starting and ending nodes coincide



Distances

- **Shortest path** (between any two nodes)

  the path with the minimum length, which is called the distance



  it is not unique!

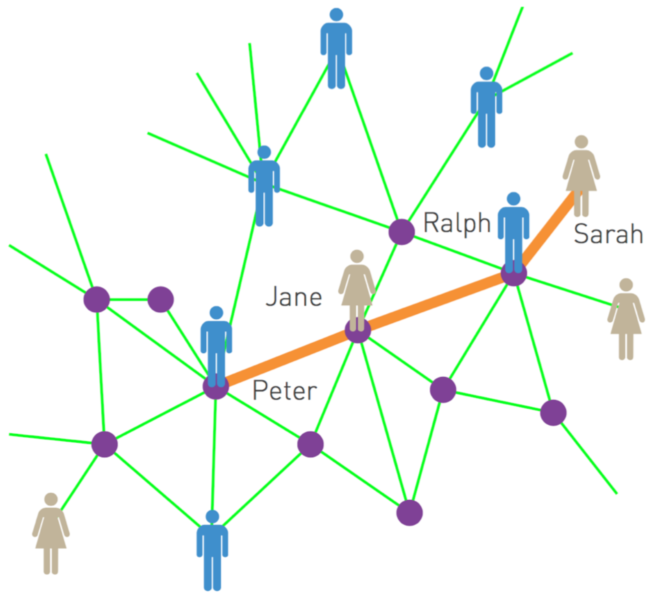
- **Diameter** (of the network)

  the highest distance in the network

Average path length

average distance between all nodes pairs (apply an algorithm to all node couples, and take the average)

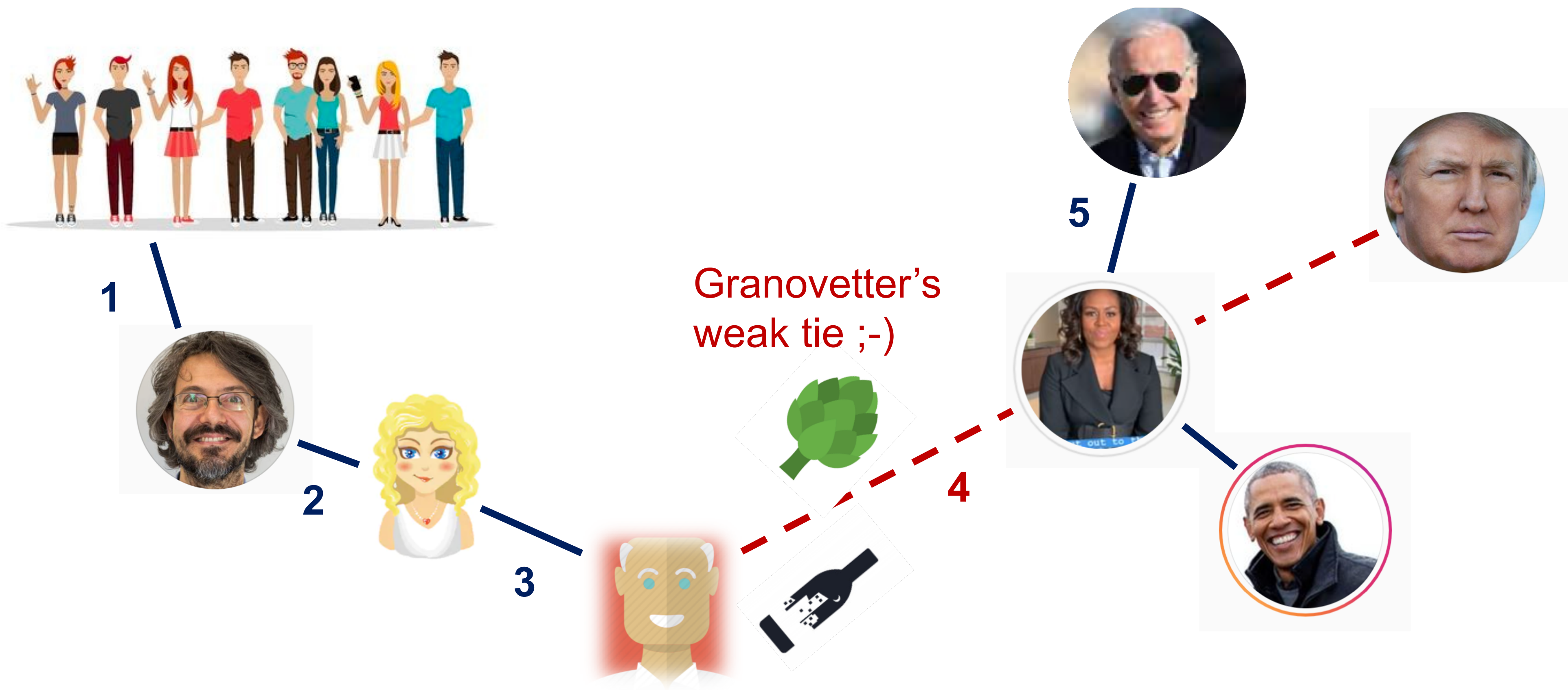
- In real networks distance between two randomly chosen nodes is generally short
- Milgram [1967]: *6 degrees of separation*



- What does this mean?
  We are more connected than we think

Small world

we and the US presidents



Connectivity

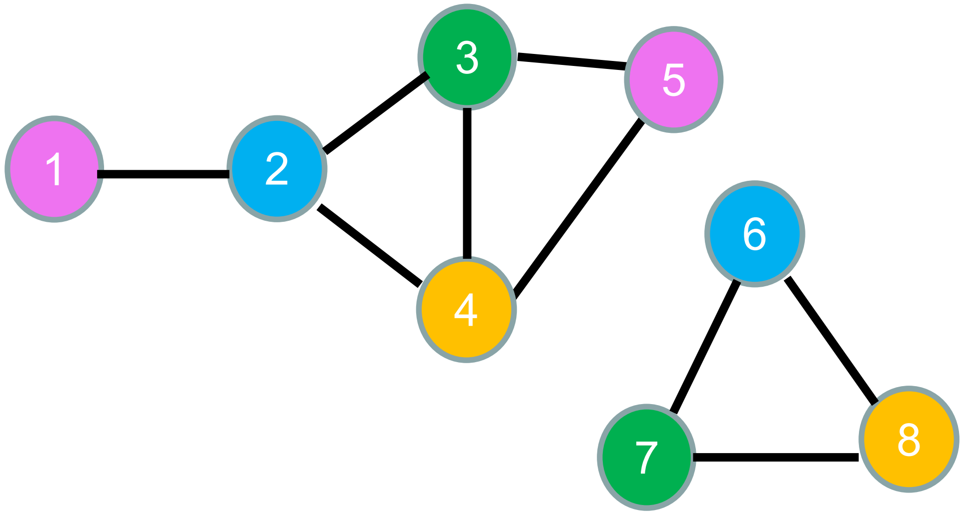
- **Connected graph (undirected)**

  for all couples (*i*,*j*) there exists a path connecting them

  if disconnected, we count the # of connected components (e.g., use BFS and iterate)

- **Giant component (the biggest one)**

- **Isolates (the other ones)**

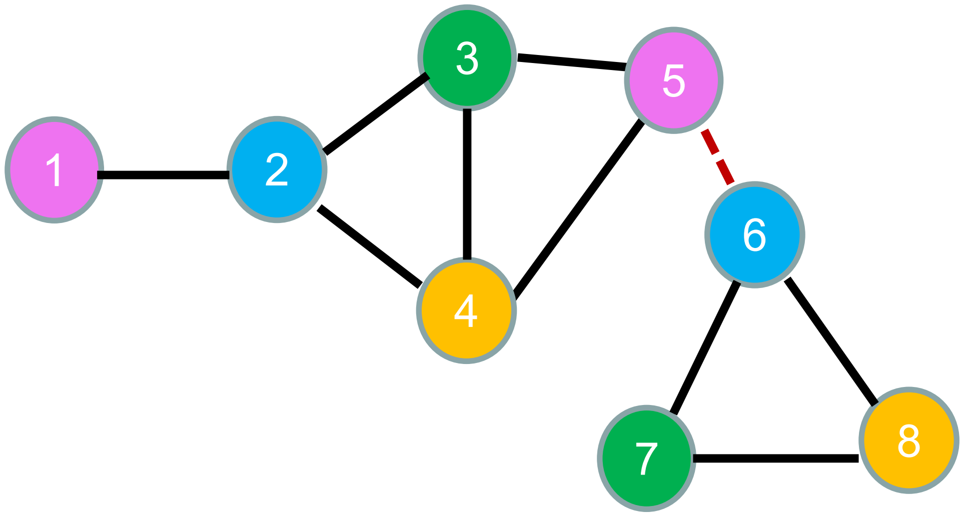


$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

block-diagonal matrix

- A **bridge** is a link between two connected components

  its removal would make the network disconnected



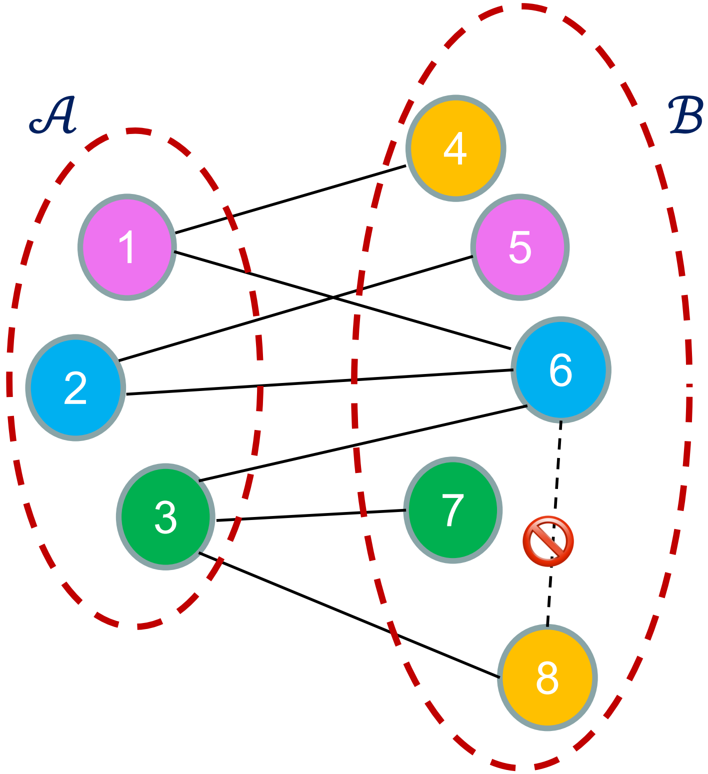
$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

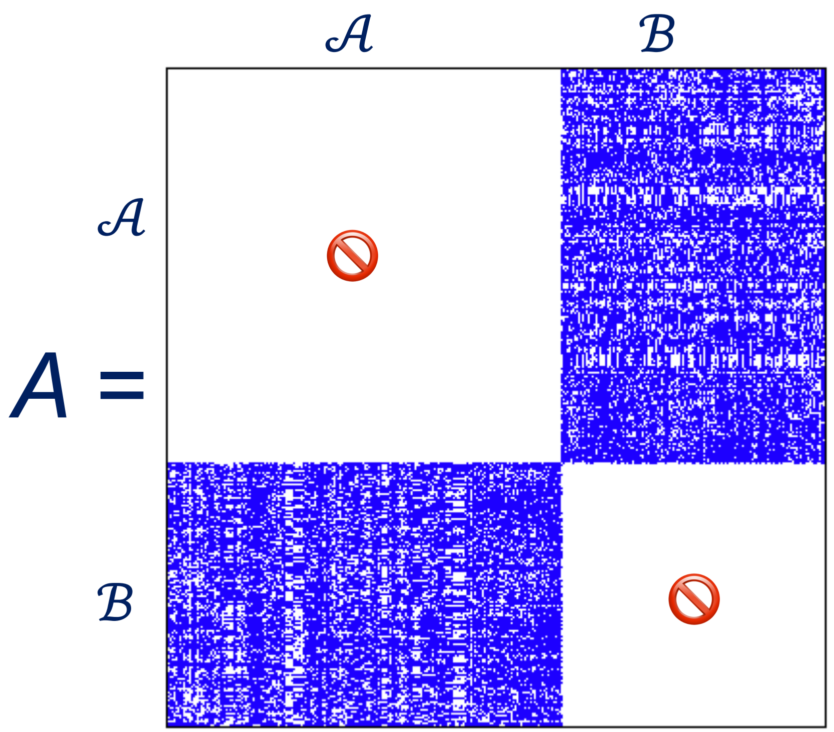
Bipartite graphs

and semantic networks

Bipartite graphs

- Connections are available only between the groups $\mathcal{A}$ and $\mathcal{B}$





Bipartite graph example

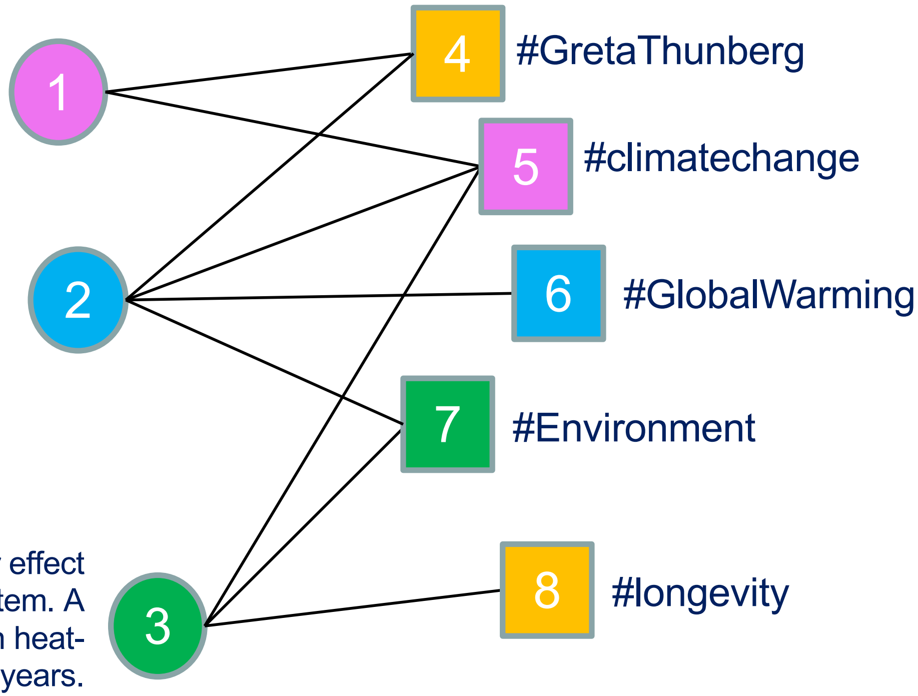
Tweets

Hashtags

those who think they are crazy enough to change the world eventually do. #climatechange #ClimateCrisis #ClimateAction #GretaThunberg #Greta

Hopefully these kids will succeed where past generations have failed. #TheResistance #FBR #ClimateChange #Environment #GlobalWarming #GretaThunberg

The #environment can have a major effect on the human cardiovascular system. A new study has found an increase in heat-induced #heartattack risk in recent years. Could #ClimateChange be a risk factor? #longevity



Meaning

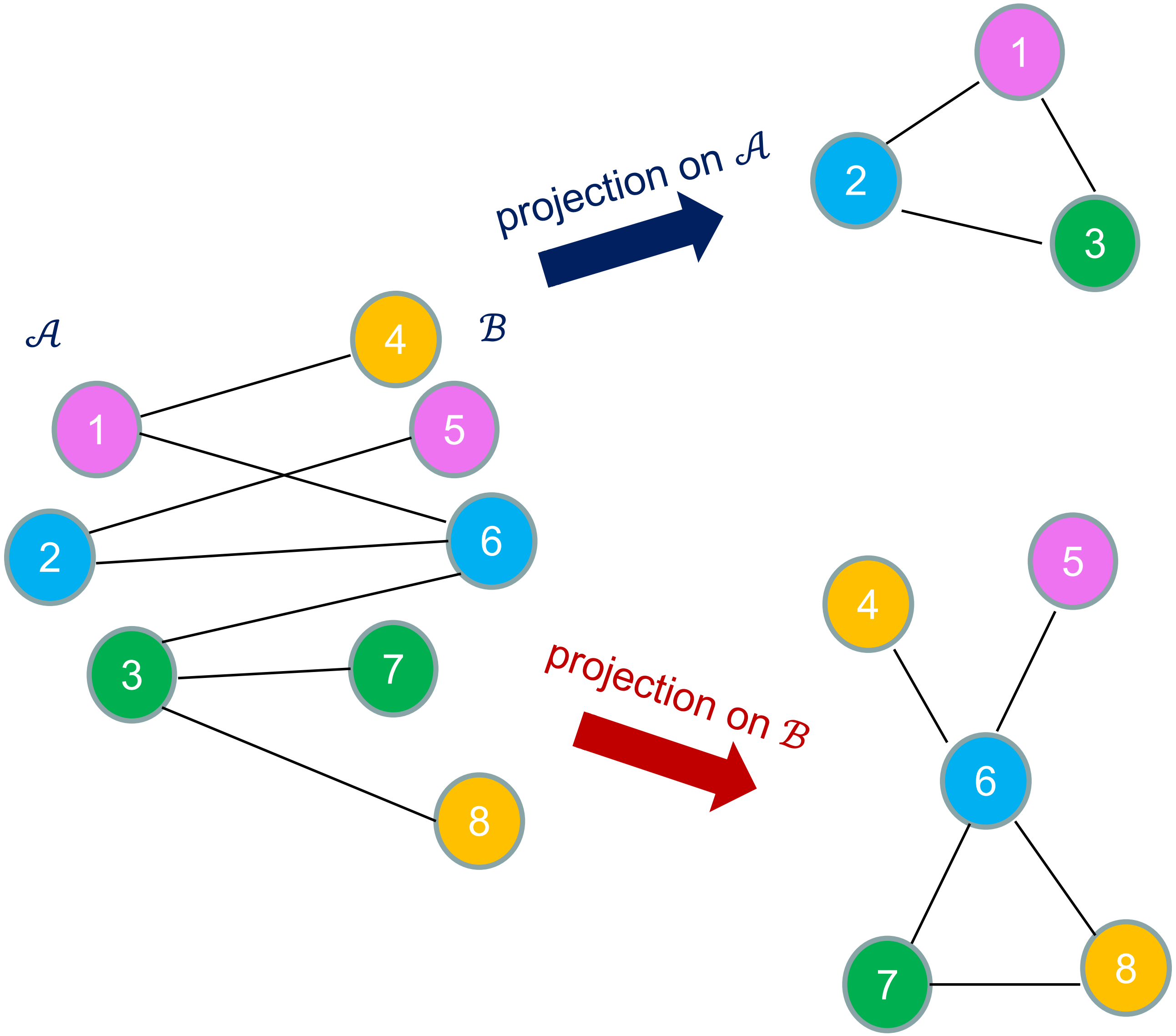
- Bipartite graphs represent memberships/relationships, e.g., groups ($\mathcal{A}$) to which people ($\mathcal{B}$) belong

  examples: movies/actors, classes/students, conferences/authors

- We can build separate networks (projections) for $\mathcal{A}$ and $\mathcal{B}$ (sometimes this is useful)

  in the movies/actors example being linked can be interpreted in two ways: "actors in the same movie" (projection on $\mathcal{B}$), or "movies sharing the same actor" (projection on $\mathcal{A}$)

Abstract example



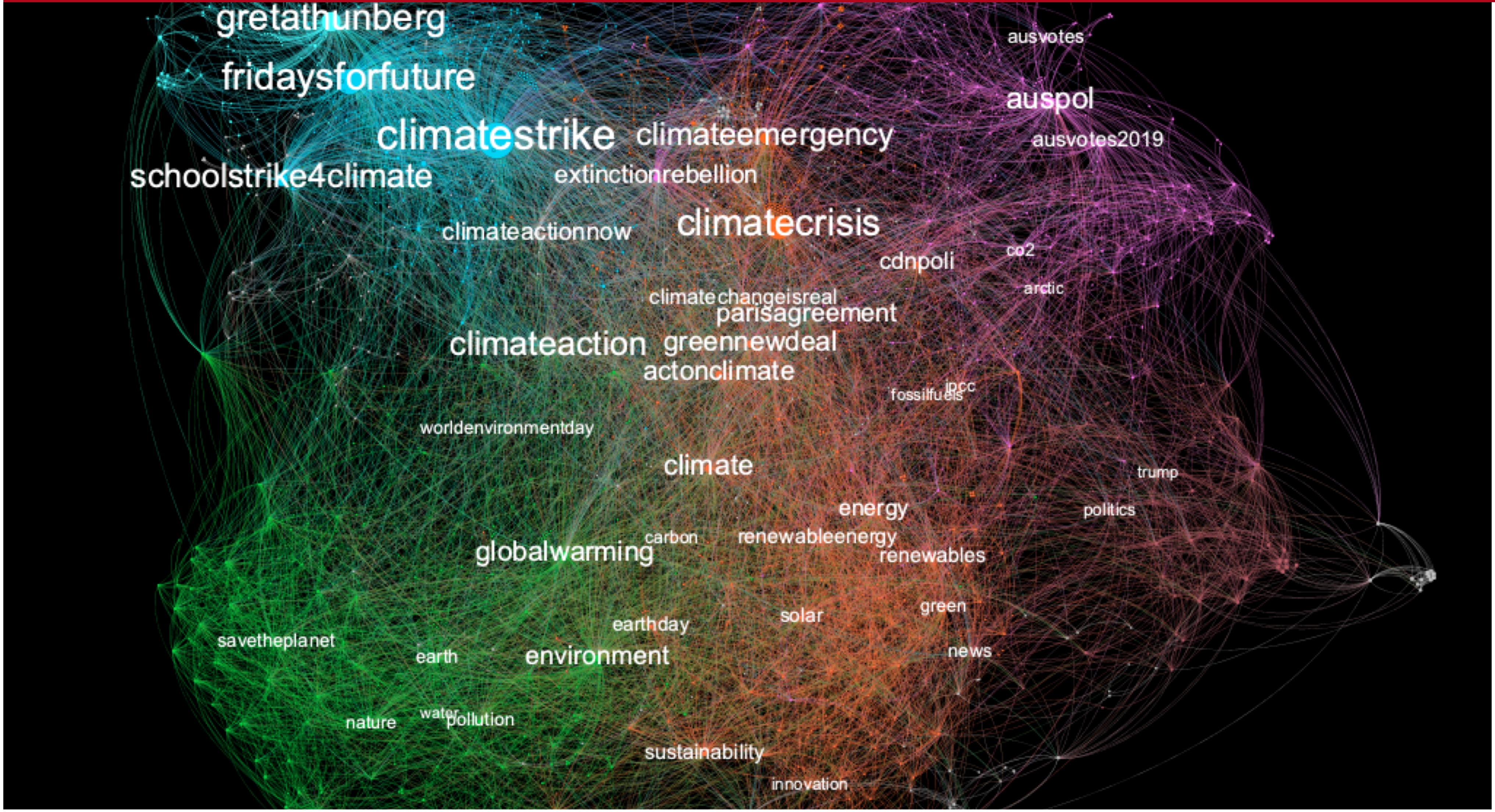
Nodes are linked if they have a common neighbour in $\mathcal{B}$

PS: we say that nodes *i* and *j* have a common neighbour *k* if both *i* and *j* are connected to *k*

Nodes are linked if they have a common neighbour in $\mathcal{A}$

Projection on a semantic network

#hashtags that appear in the same tweet are linked



#climateaction tweets after Greta Thunberg

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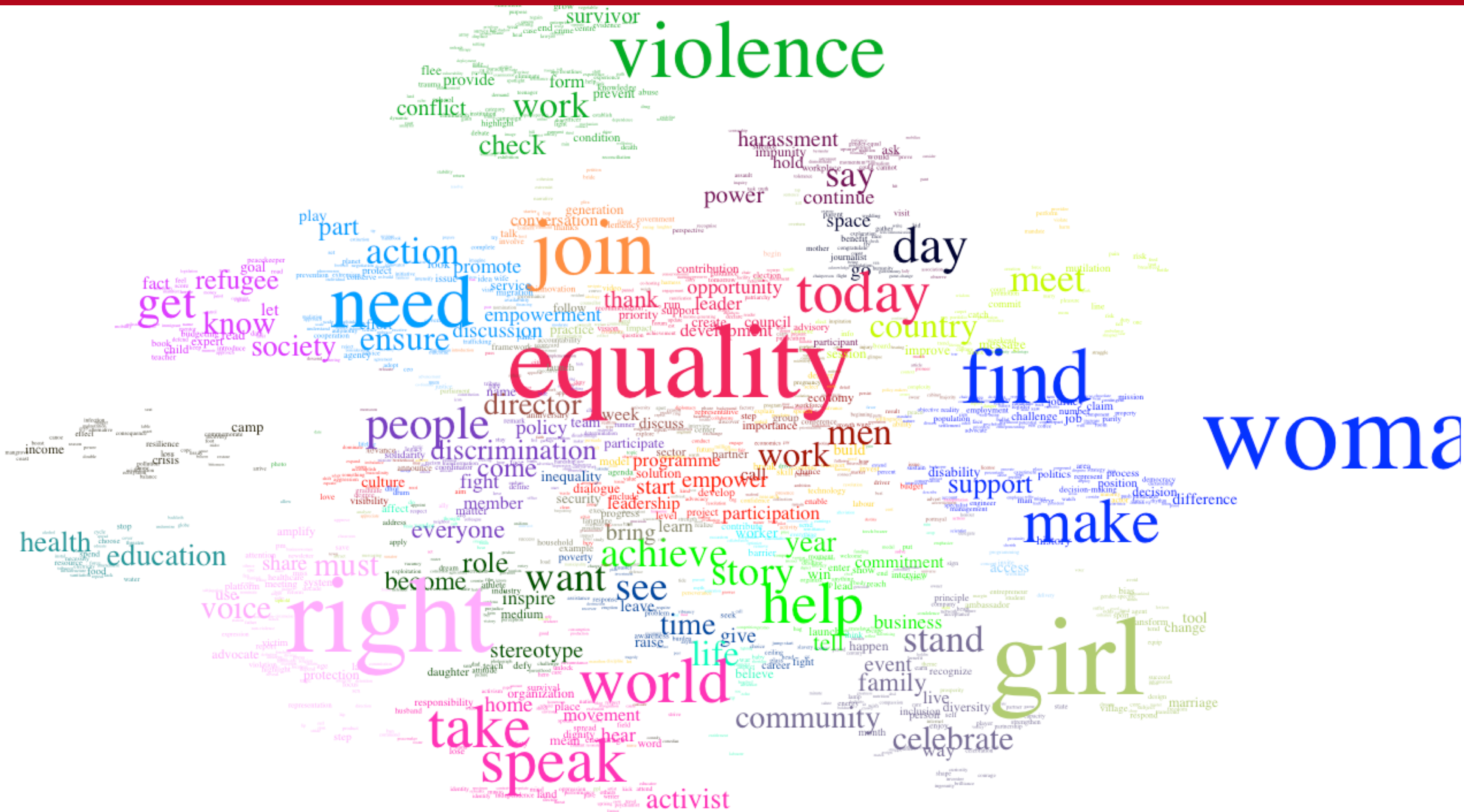
Projection on a semantic network

words that appear in the same tweet are linked



Takeaways so far

- (un)Directed graphs
- Weighted and signed graphs
- Adjacency matrix & edge list
- Distances
- Giant component, isolates, bridges
- Bipartite graphs & projections

Degree centrality

a first approach to node importance

Centrality

From Wikipedia, the free encyclopedia

For the statistical concept, see Central tendency.

In graph theory and network analysis, indicators of **centrality** identify the most important vertices within a graph. Applications include identifying the most influential person(s) in a social network, key infrastructure nodes in the Internet or urban networks, and super-spreaders of disease. Centrality concepts were first developed in social network analysis, and many of the terms used to measure centrality reflect their sociological origin.[1] They should not be confused with node influence metrics, which seek to quantify the influence of every node in the network.

Degree centrality [edit]

Main article: Degree (graph theory)

PageRank centrality [edit]

Main article: PageRank

Betweenness centrality [edit]

Main article: Betweenness centrality

Eigenvector centrality [edit]

Main article: Eigenvector centrality

Closeness centrality [edit]

Main article: Closeness centrality

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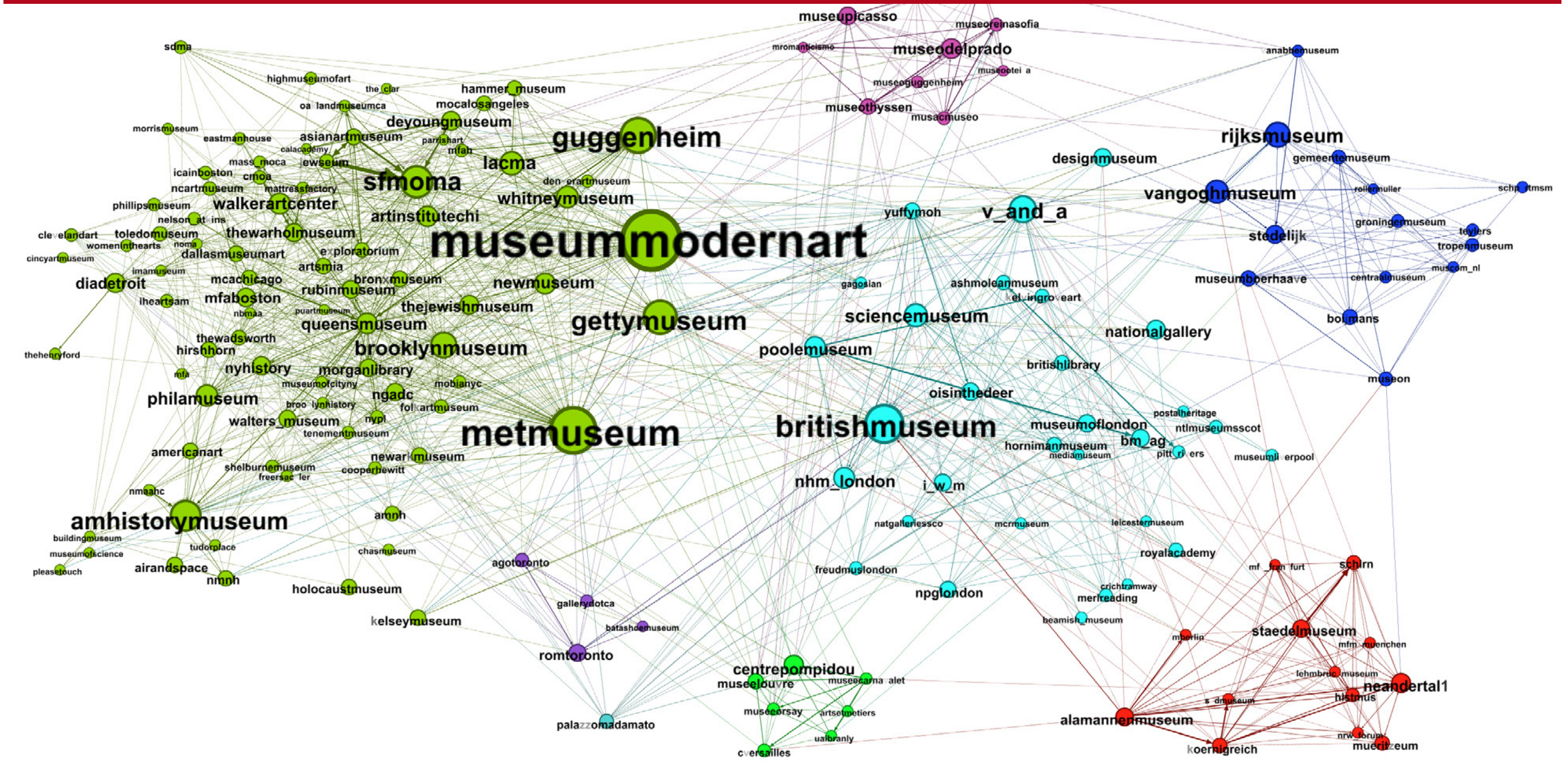
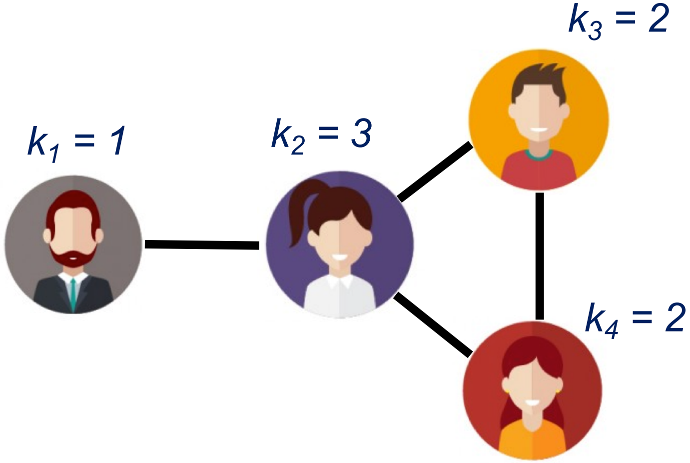
An example of node centrality

museums network



- The degree k_i of node i in an undirected networks is

  the # of links i has to other nodes, or
  the # of nodes i is linked to



The average degree is

$$\langle k \rangle = \sum_i k_i / N = (1+3+2+2)/4 = 2$$

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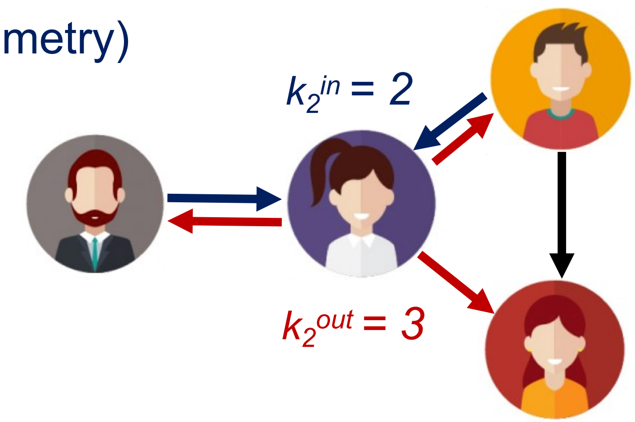
Node degree
directed networks

- For directed networks we distinguish between

  in-degree k_i^{in} = # of entering links

  out-degree k_i^{out}= # of exiting links

  (undirected: $k_i^{in} = k_i^{out}$ due to the symmetry)



The average degree is

$$\langle k \rangle = \sum k_i^{out} / N = (1+3+2+0)/4$$
$$= \sum k_i^{in} / N = (1+2+1+2)/4$$
$$= 3/2$$

Meaning

- A social-capital measure of cohesion
- In-degree = importance as an Authority
- Out-degree = importance as a Hub

an influencer:
authority or hub?

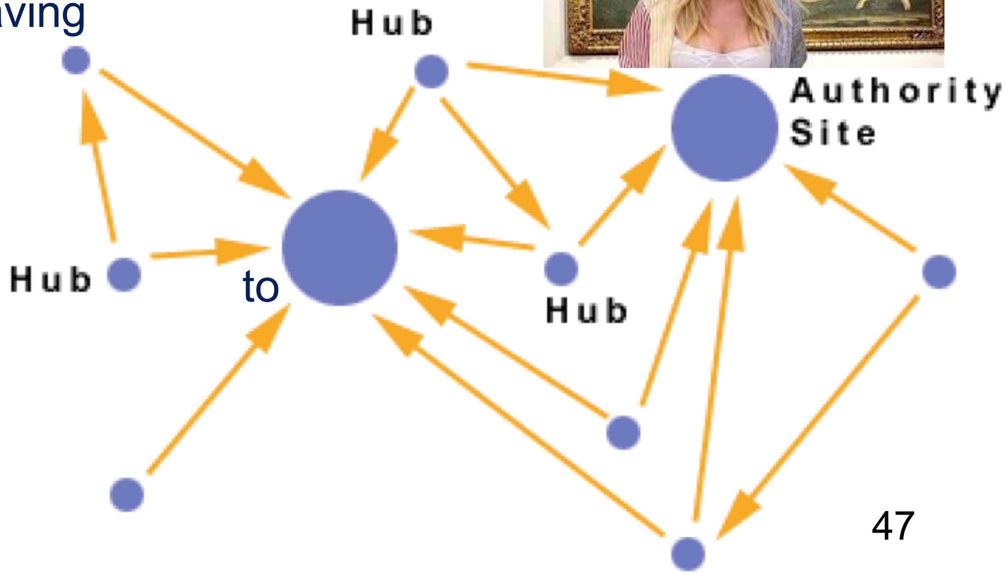
In www:

- Authorities (quality as a content provider)

nodes that contain useful information, or having a high number of edges pointing to them (e.g., course homepages)

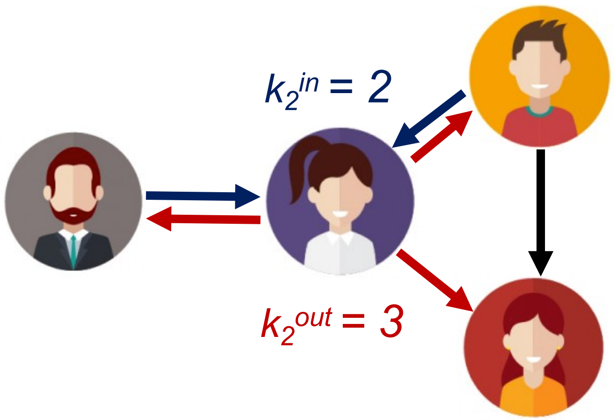
- Hubs (quality as an expert)

trustworthy nodes, or nodes that link many authorities (e.g., course bulletin)



Adjacency matrix and degree

- The in (out) degree can be obtained by summing the adjacency matrix over rows (columns)



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$k_2^{in} = 2$ (row 2)

$k_2^{out} = 3$ (column 2)

Real networks are sparse

- The maximum degree is *N-1*
- In real networks *<k>* ≪ *N-1*

NETWORK	N	L	$\langle k \rangle$
Internet	192,244	609,066	6.34
WWW	325,729	1,497,134	4.60
Mobile Phone Calls	36,595	91,826	2.51
Email	57,194	103,731	1.81
Science Collaboration	23,133	93,439	8.08
Actor Network	702,388	29,397,908	83.71
Citation Network	449,673	4,689,479	10.43

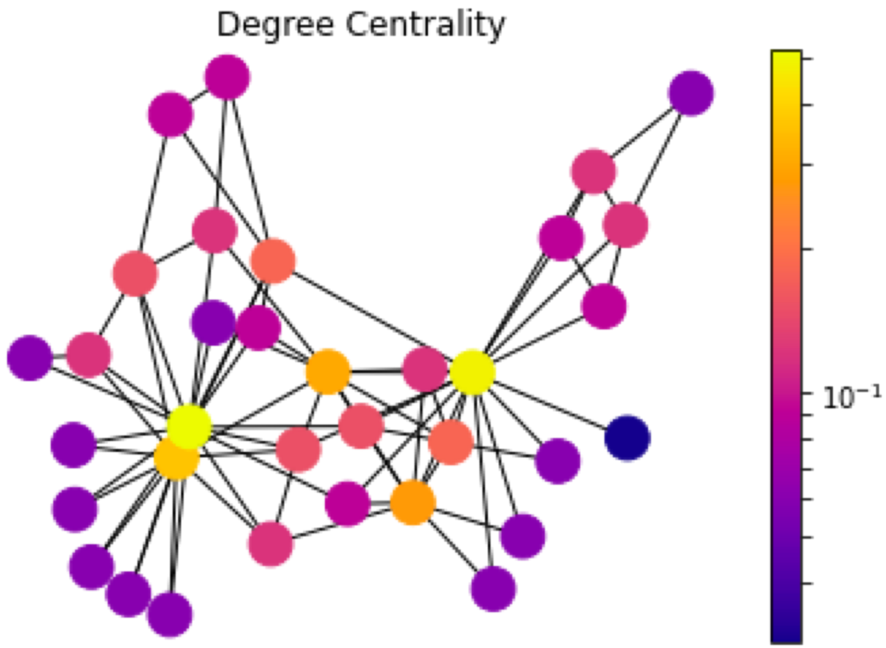
Visualizing degree centrality

how to get useful insights on centrality

Graphical representations
of degree centrality

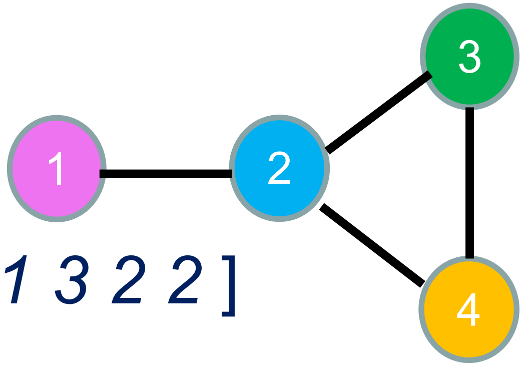
by size

by colour

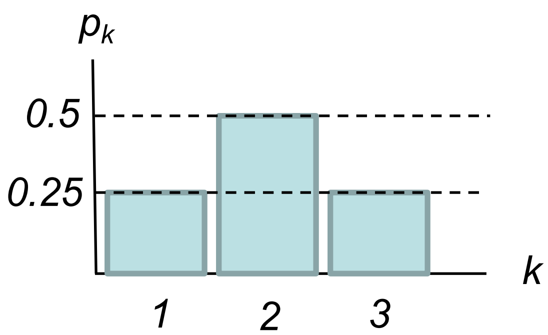


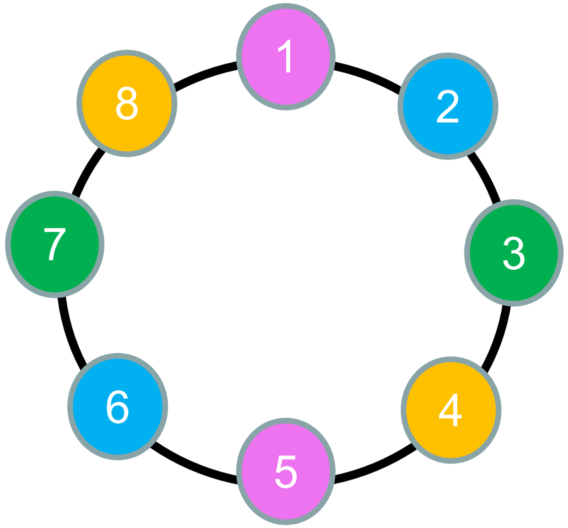
Degree distribution

- a <u>probability</u> distribution p_k
- p_k = the fraction of nodes that have degree equal to k
- p_k = # of nodes with degree k, divided by N

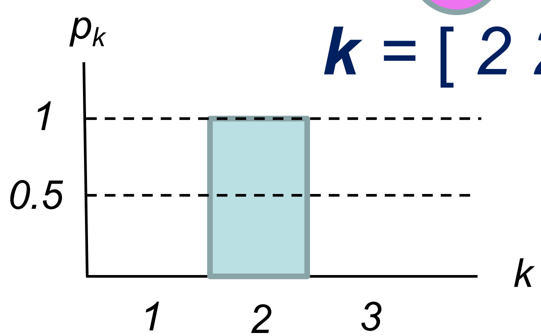


$\mathbf{k}$ = [*1 3 2 2*]



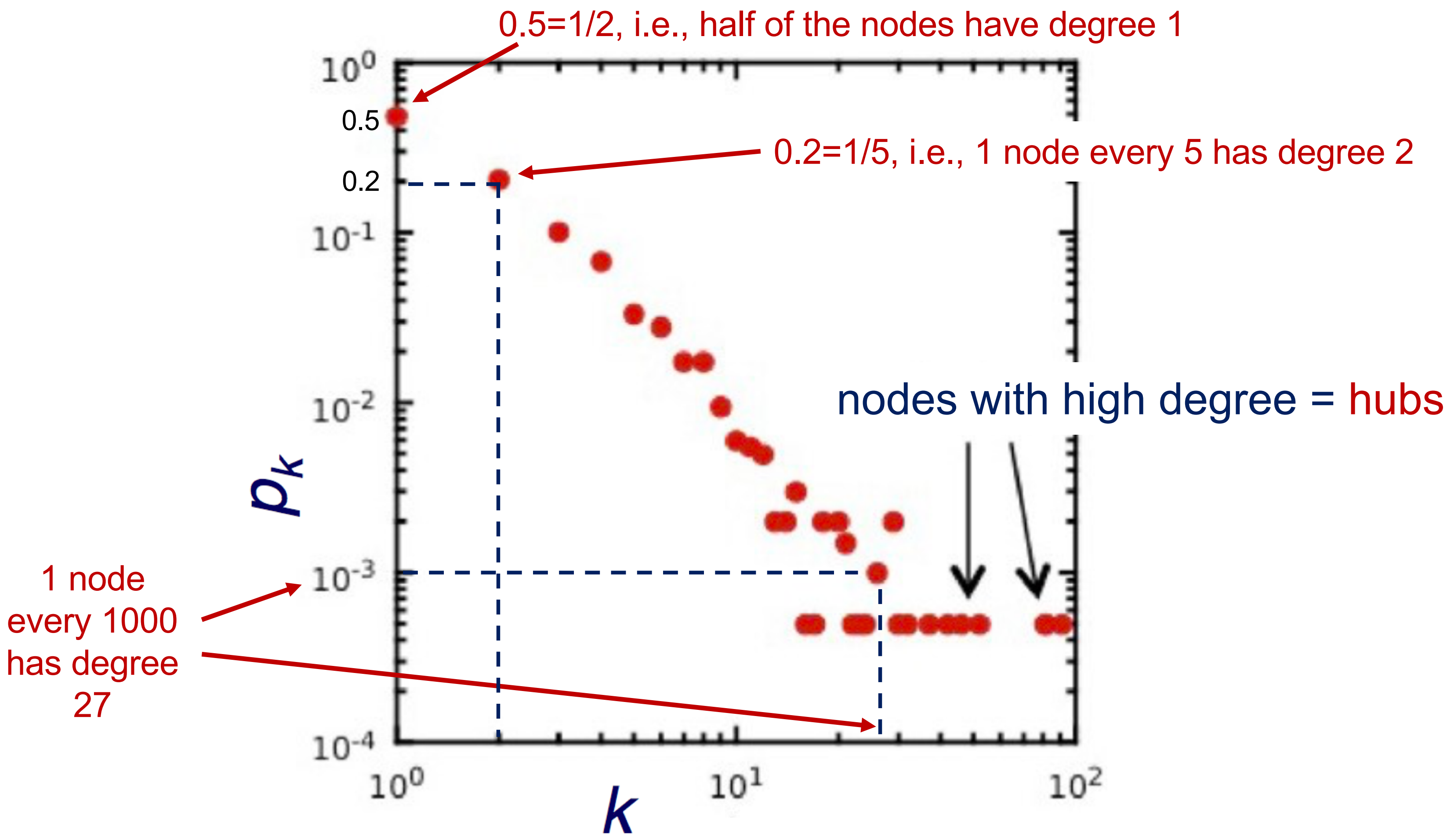


$\mathbf{k}$ = [2 2 2 2 2 2 2 2]



Log-log plot

- In real (large) networks, degrees have a large range → log representation

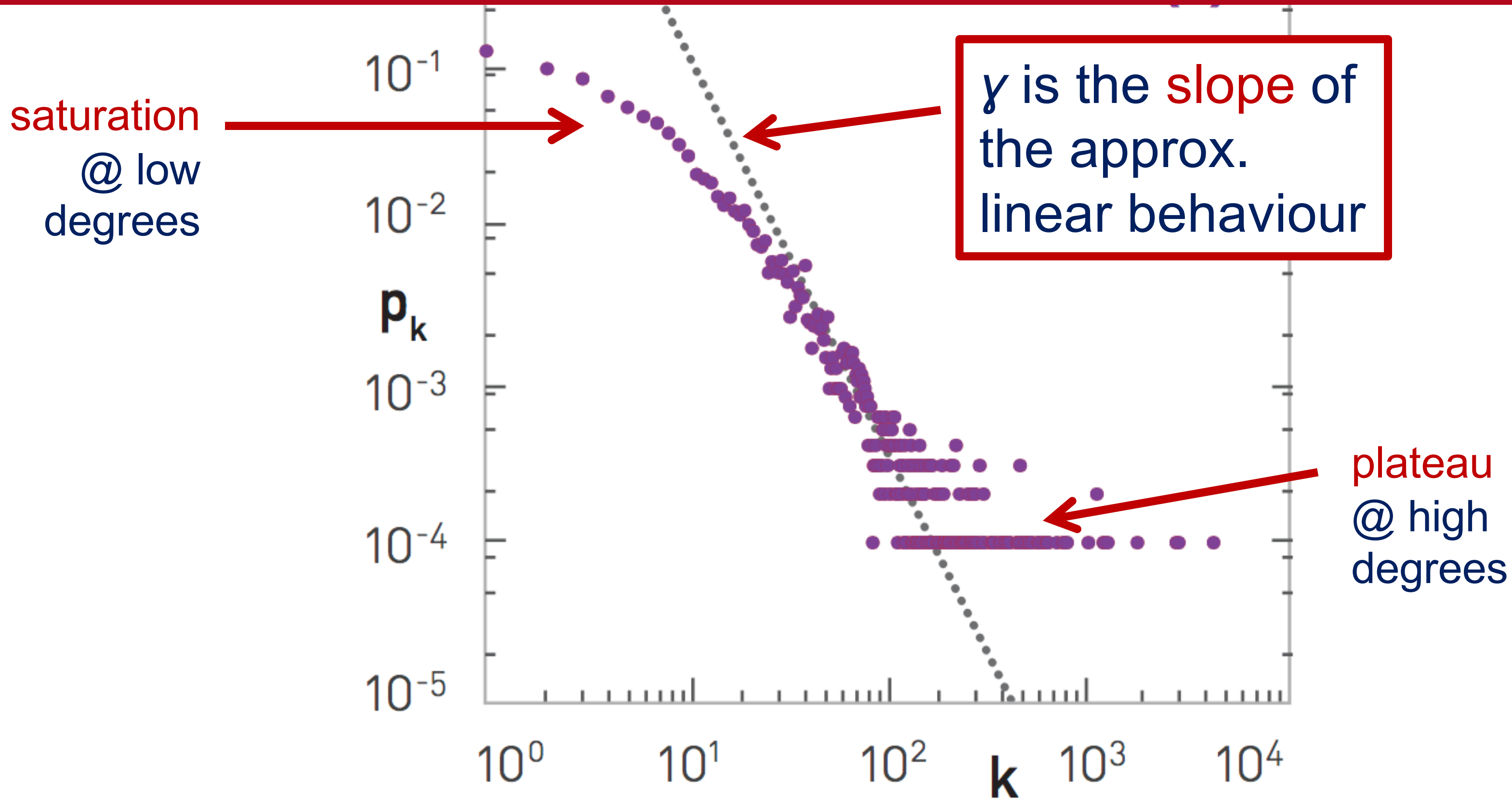


Scale-free networks

those that follow a power-law

The power law

typical of social networks



Why the name power-law? Because the (approx.) linear behaviour in the log domain ensures

$$\ln(p_k) = c - \gamma \cdot \ln(k) \quad \rightarrow \quad p_k = C\,k^{-\gamma}$$

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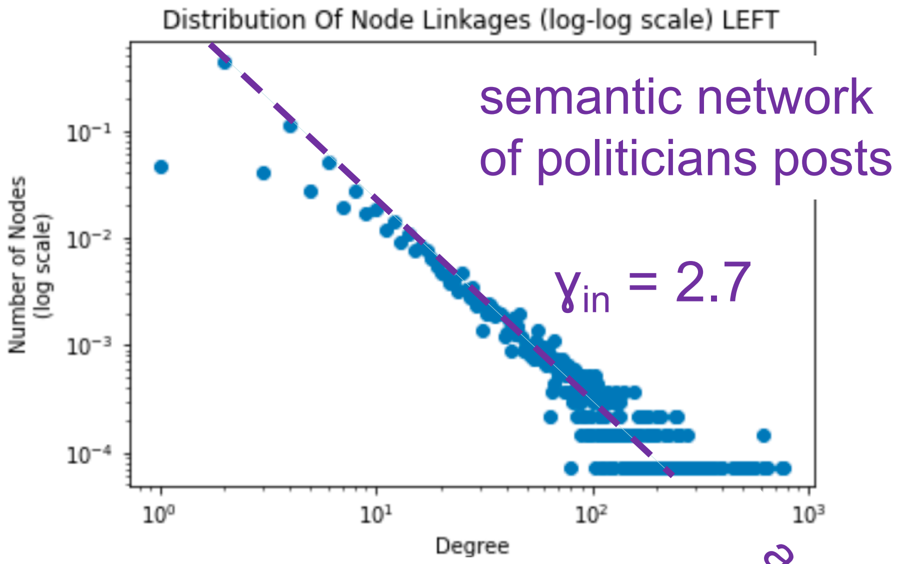
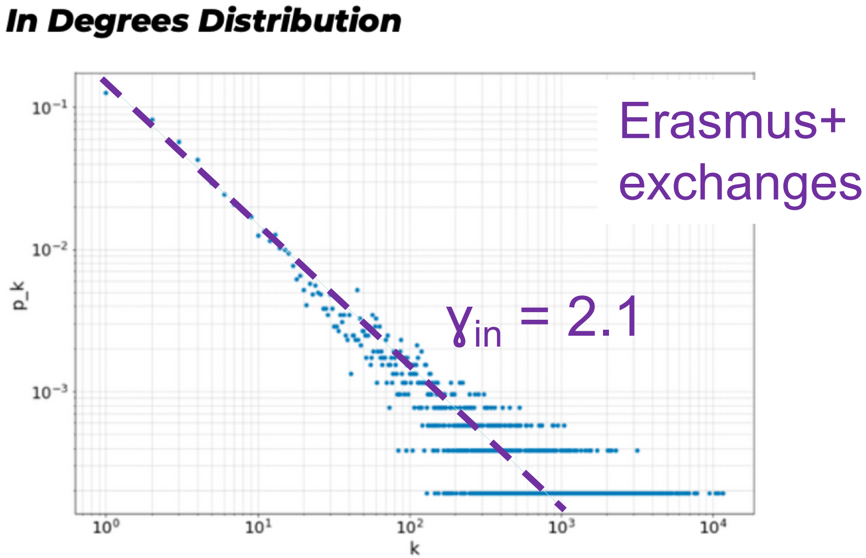
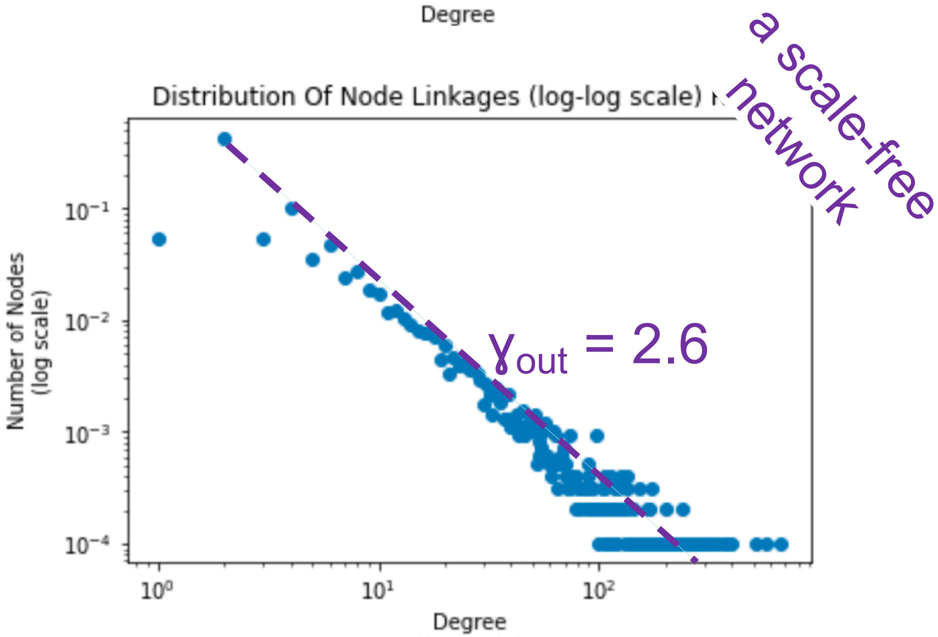
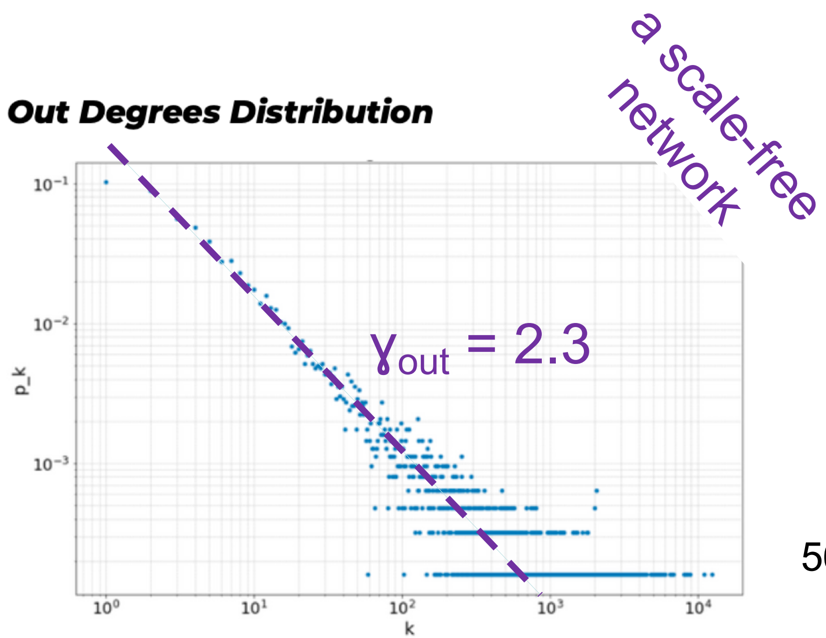
Examples
from past projects









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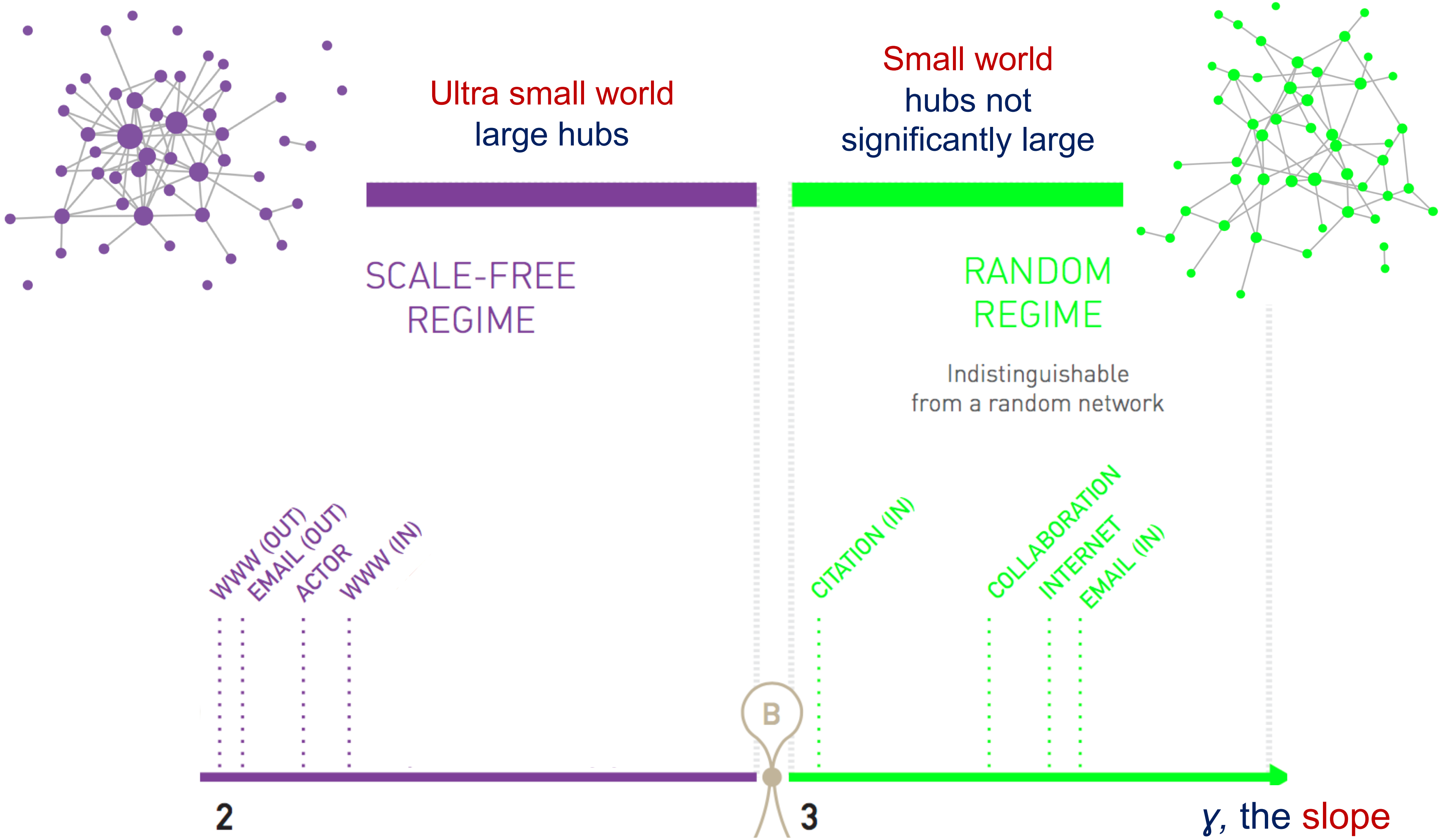
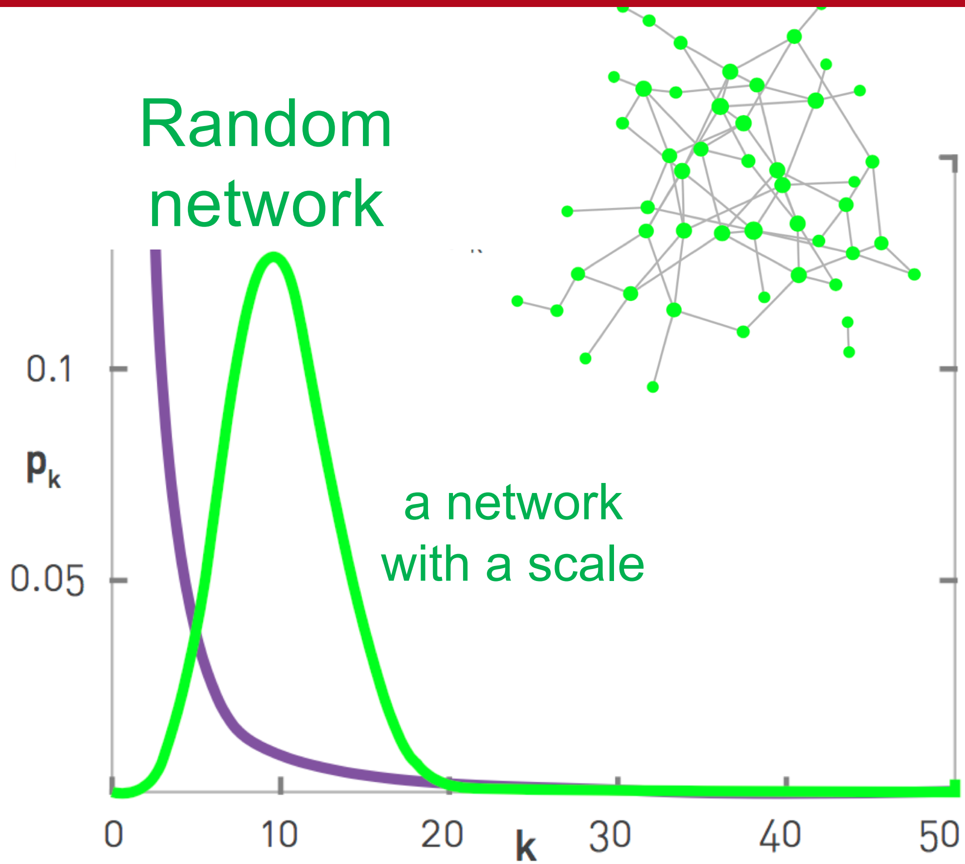
The ultra-small-world
of scale-free networks



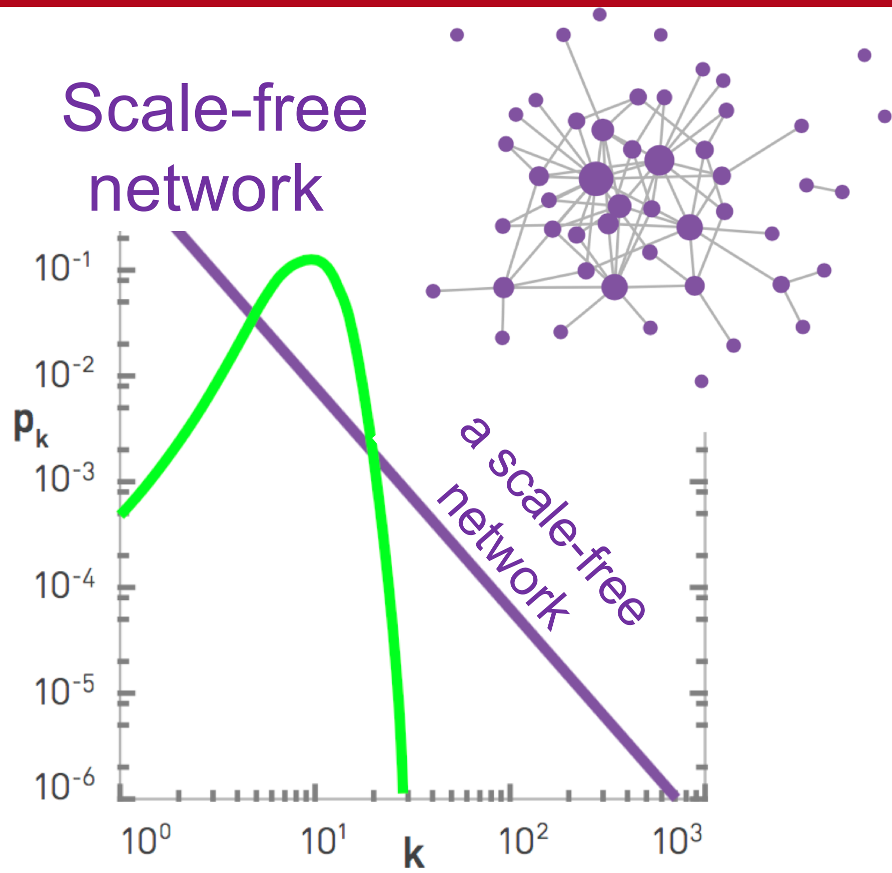
Scale-free networks
versus random networks

Random network



- Randomly wired network
- Has smaller hubs
- Needs a linear plot

Scale-free network

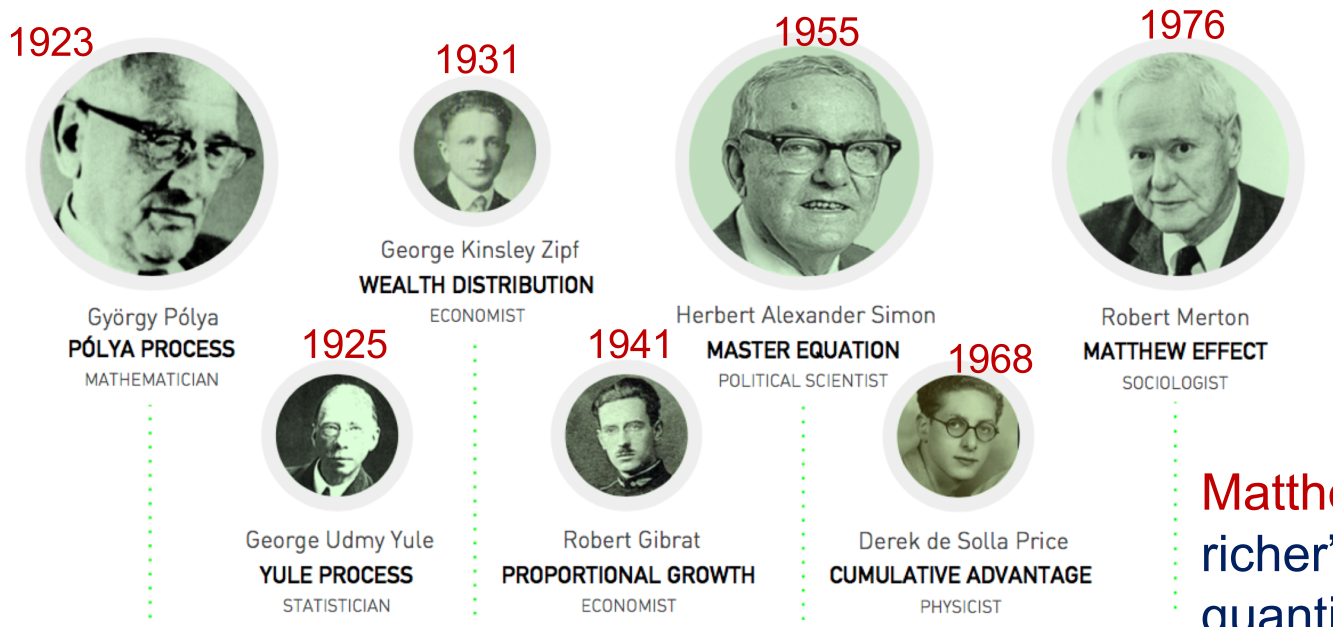


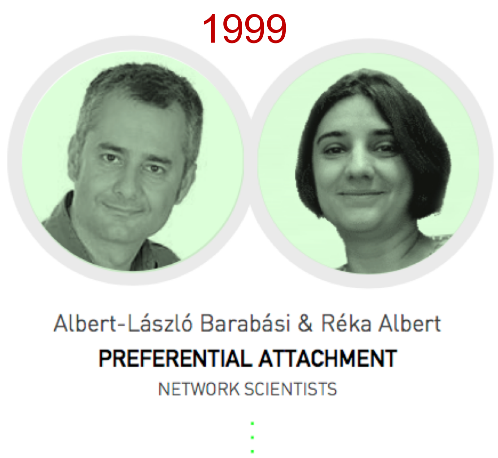
- Power-law network
- Has big hubs
- Needs a log-log plot

Nodes link to the more connected nodes

e.g., think of www

This idea has a long history





Matthew effect: “rich gets richer”, i.e., high connectivity quantifies attractiveness

The copying model
explaining preferential attachment

- **Citation network**

  researchers decide what papers to read and cite by "copying" references from papers they have read → papers with more citations are more likely to be cited

- **Social network**

  the more acquaintances an individual has, the higher the chancer of getting new friends, i.e., we "copy" the friends of friends → difficult to get friends if you have none

- **Semantic network**

  does the model apply here?

Attractiveness

a further essential concept to explain the power-law

- There is an innate ability of a node to attract links
  just a quality assessment of the individual

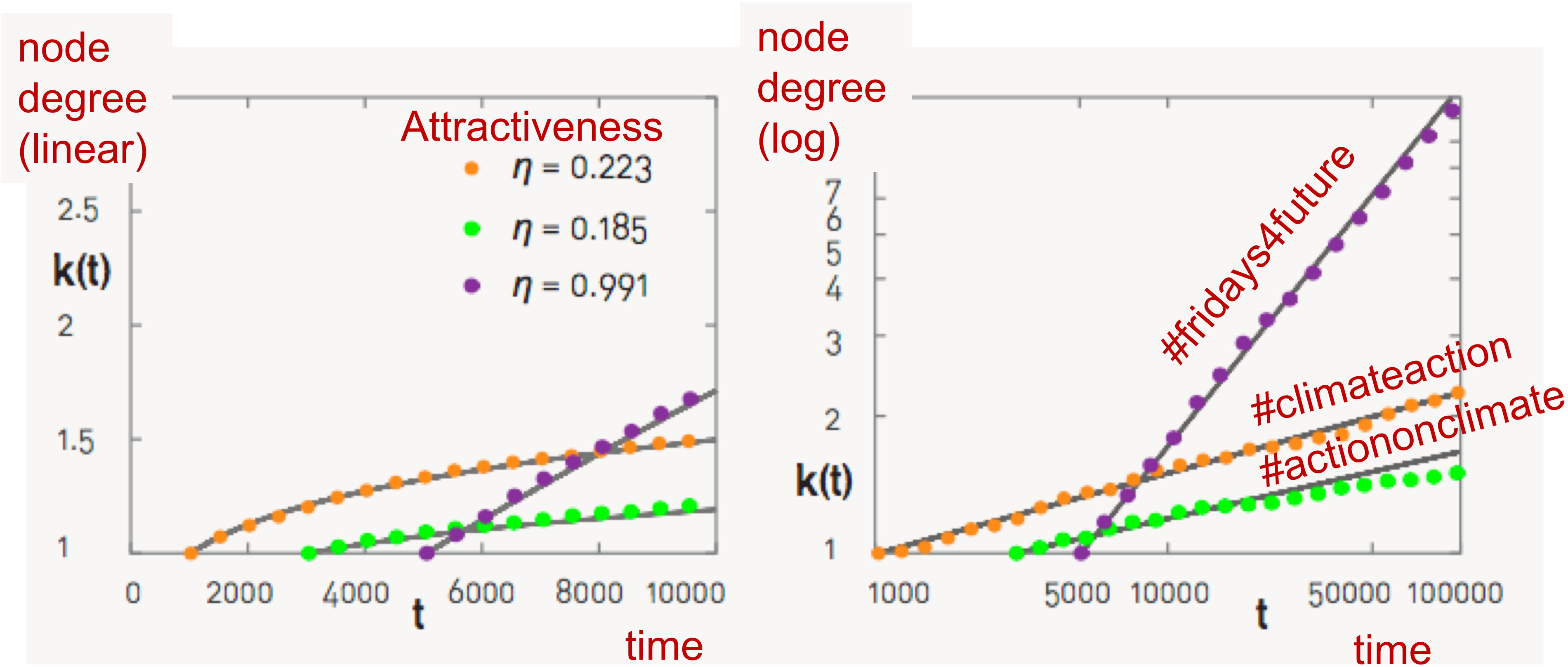
- Otherwise oldest nodes would have an inherent advantage and cannot be defeated (*first mover's advantage*), which is in contrast with intuition and evidence

  e.g., Altavista [1990] → Google [2000] → Facebook [2011] → Instagram [202?]

  e.g., #parisagreement [2018] → #fridays4future [2019]

Attractiveness

a visual example



η_i can be measured by data scientists !

Takeaways

- Degree, degree distribution, loglog plot
- Authorities and hubs
- Power law, scale-free networks
- Slope, Ultra-small-world regime
- Preferential attachment
- Attractiveness